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Non-Local Structure of Field Theories with Non-Renormalizable Interaction (*).

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Summary. — Quantized field theories of the second kind, the so-called non-renormalizable theories, are investigated. It has recently been shown that a number of those theories actually can be renormalized. We show that field theories of the second kind, pertaining to derivative couplings, possess non-local structure. The full recoil neutral PS-PV theory in its non-linear form is discussed. The non-localizability of both the unrenormalized and the renormalized theories is due to the fact that the infinite set of topologically independent direct interactions, which characterizes PS-PV theory, is equivalent to a new type of indirect interaction. The non-localizability manifests itself in an indeterminacy in space-time of the light-cone. This indeterminacy can be interpreted in terms of a statistical spread in space-time of the positions of two interacting point-nucleons over a space-time volume the size of which is determined by the coupling constant $|g|$ of the linear version of PS-PV theory. The length $|g|$ loses its meaning as a coupling in the non-linear formalism in favour of the nucleon mass and plays now the role of a fundamental structure constant. In particular, $|g|$ determines the region in which the equivalence theorem and the $1/r^3$-potential become invalid and repulsive-core potentials begin to act. In virtue of the oscillatory behaviour of the non-tempered momentum space operators in the high energy region, positive-definiteness conditions can be established only for regions $|k_\nu^2| \lesssim |1/g^2|$ and $|x_\nu^2| \lesssim |g^2|$. The non-localizability is concentrated in these domains and $|g|$ seems to play the role of the meson Compton wave length. A new interpretation of the non-linear PS-PV theory is sug-

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gested. It is conjectured that also β interactions give rise to non-local structures. Multiple propagators are introduced to render the formal coupling constant expansions in theories of the second kind mathematically meaningful without producing non-renormalizable infinities. Arguments are advanced against the mathematical consistency of the concept of creation (annihilation) and localizability of more than one point-particle at a single space-time point.

1. – Introduction.

This is the first of a series of papers devoted to a study of problems of quantized field theories of the second kind. Meson theories with derivative couplings, in particular, pseudoscalar (PS) meson theory with pseudovector (PV) coupling, and direct interactions such as Fermi couplings belong to this class. Those interactions, although physically attractive, have been neglected in recent years because they seemed to be non-renormalizable within the frame of customary perturbation analysis. This argument against field theories of the 2-nd kind has recently been disproved by ARNOWITT and DESER [1] and by COOPER [2] who have shown that a large number, if not all, of those theories actually can be renormalized. The importance of field theories of the second kind in the theory of elementary particles, in particular, in the high energy and small space-time regions, has often been stressed by HEISENBERG [3] who conjectured the introduction of a fundamental length through the intermediary of those interactions.

In Sect. **2** of the present paper we discuss briefly the main differences between interactions of the 1-st and 2-nd kind originating from the different topological structure of the respective sets of S-matrix elements. The special object of our attention is the neutral PS-PV theory in its non-linear version. The S-matrix formalism for this interaction of the second kind can be formulated in terms of the usual coupling constant expansion if one presupposes the mathematical existence of multiple propagators such as $(\Delta_F)^n$, $(S_F)^n$. The fundamental equations of LEHMANN, SYMANZIK and ZIMMERMANN [4], applied to field theories of the second kind, can likewise be solved in terms of those multiple propagators.

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(1) R. ARNOWITT and S. DESER: *Phys. Rev.*, **100**, 349 (1955); referred to as AD.

(2) L. N. COOPER: *Phys. Rev.*, **100**, 362 (1955). Thanks are due to Dr. COOPER for communications in advance of publication.

(3) W. HEISENBERG: *Ann. Phys.*, **32**, 20 (1938); *Nachr. Gött. Akad. Wiss., Fests. 1951.*

(4) H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **1**, 205 (1955); **6**, 319 (1957); referred to as LSZ.

In Sect. 3 we discuss from a general point of view the problem of the mathematical and physical consistency of the concept of multiple creation (annihilation) of point-particles at a single point of the mathematical, flat space-time continuum. In virtue of a recent mathematical theorem, due to KÖNIG (5), we conclude that the strict localizability of more than one point-like event at one single point of Minkowski space is incompatible with some basic postulates of present quantized field theories, in particular, with causal propagation of those events. These arguments would seem to apply to both theories of the first and second kind. We show in Sect. 4 that it is still possible, within the frame of exact mathematics, to define multiple propagators of the type $(\Delta_F)^n$, $(S_F)^n$. However, this goal can be reached if and only if one renounces the unrestricted validity of a set of mathematical axioms. In particular, in a vector space in which multiple propagators are defined, the existence of a continuous, infinitesimal displacement operator is necessarily incompatible with the postulated invariance of field theories under similarity transformations.

In Sect. 5 we shall show that the topological structure of the S-matrix elements of field theories with derivative couplings, exemplified by neutral PS-PV theory, imparts to those theories a non-local, say, a global structure. This result, however, is *not* a consequence of the arguments advanced in Sect. 3 against the consistency of the concept of direct relativistic (contact) interaction. Instead, it is a consequence of the intervention of infinite series of derivatives of δ-functions which make up part of the nucleon propagators. The particular series encountered with represent non-local quantities of a rather unfamiliar type and determine the global structure of the theory in the same sense as, for example, the δ terms in Δ_F, S_F characterize the local structure of theories of the 1-st kind. The non-localizability of theories with derivative couplings may likewise be attributed to the fact that it is mathematically impossible to localize an infinite number of point-particles at one single point of Minkowski space. However, this result is completely independent of the results of Sect 3. mentioned above.

The non-localizability of interacting point-nucleons, in the frame of PS-PV theory, has nothing in common with the familiar non-local concepts associated with a structure function. No such quantity is encountered with here. Instead, the non-local structure of the theory manifests itself in an indeterminacy in space-time of the light-cone which requires a statistical interpretation. In consequence of this indeterminacy, the positions of two interacting point-nucleons spread over a space-time domain whose size is determined by the coupling constant g (dimension of a length) of the linear version of PS-PV theory. We may say that the non-localizability, encountered with in both

(5) H. KÖNIG: *Arch. Math.*, **6**, 391 (1955).

the unrenormalized and the renormalized formalisms, is due to the fact that the infinite set of topologically independent direct interactions, which characterizes the actually non-linear PS-PV theory, is equivalent to a new type of indirect interaction.

We shall show that the coupling constant of the linear representation of PS-PV theory, *viz.* the length g ($g = \sqrt{4\pi} f/\mu$ in familiar notation), loses entirely its meaning as a coupling in favor of the nucleon mass, the latter describing now the strength of the nucleon-nucleon interaction. Instead, the length g describes now an inner mesonic structure of the interacting nucleons and governs the indeterminacy of the light-cone mentioned above. In virtue of the experimental result $f^2 \approx 0.08$ we conclude that $|g|$ is of the order of the meson Compton wave length, $|g| \approx 1/\mu$, and this is just the mean distance in space-time (space- and time-like) over which the positions of the interacting point-nucleons spread. It is therefore not surprising that the meson mass parameter involved in the formalism via the Δ_F propagator loses its meaning, at least partially, in favor of $1/|g|$. The spread in space-time is essentially independent of the value of the parameter μ^* in Δ_F. These facts lead ultimately to a new interpretation of PS-PV theory.

The parameter $|g|$ will be seen to act as a critical length. This length determines not only the indeterminacy in space-time of the light-cone but determines likewise the region in which Dyson's equivalence theorem and the familiar $1/r^3$-potential become invalid and repulsive « potentials » come into play. Wightman's ([6]) positive-definiteness conditions cannot longer be ensured in the region $|k_\nu^2| \gtrsim |1/g^2|$ but may be preserved in the domain $|k_\nu^2| < |1/g^2|$. Thus the inevitable acausal effects are restricted to the small space-time regions. These questions are briefly discussed in Sect. 6 together with a short exposition of the renormalization problem.

These results are valid for all derivative coupling theories . The associated momentum space propagators are non-tempered distributions with order of growth ([7]) ϱ, $0 < \varrho < 1$. The methods used in the discussion of derivative coupling theories apply likewise to β interactions. The model of scalar β coupling discussed by ARNOWITT and DESER ([1]) would seem to lead to momentum space propagators which also possess order of growth $\varrho > 0$, the latter property being the definitive criterion for the non-localizability of those interactions ([8]).

([6]) A. S. WIGHTMAN: *Phys. Rev.*, **101**, 860 (1956); W. SCHMIDT and K. BAUMANN: *Nuovo Cimento*, **4**, 860 (1956). The author is grateful to Dr. WIGHTMAN for a preprint.

([7]) The order of growth ϱ of an integral analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $\max_{|z| \leqslant r} |f(z)| \leqslant \exp[r^\varrho]$, $r \to \infty$, is given by $1/\varrho = \lim_{n \to \infty} [\log(1/|a_n|)/n \log n]$.

([8]) The localizability of the weak interactions may be checked by an experimental verification of Bogoljubov's dispersion relations: N. N. BOGOLJUBOV, S. M. BILENKIJ

In this paper we consider the field theories of the second kind as forming closed systems. If those interactions play a role in physics—and it seems likely that they do so—their non-localizability obviously will also influence the supposed localizability of 1-st kind interactions on account of the mutual coupling between all real particles. We remark incidentally that the IMR theorem (9) determines pion-nucleon interaction to be PS-PV, this interaction also being favoured by multiple meson production. However, whether pion-nucleon interaction actually is PS-PV is still an open question. It is worth mentioning that the generalized equivalence theorem (between PS-PS and PS-PV theory) established by AD tells us that both types of couplings behave essentially different only in small space-time domains ($|x_\nu^2| \lesssim |g^2|$) Since, as we shall see, PS-PV theory actually describes a scheme which is essentially different from the usual meson-nucleon coupling concept—it seems rather to describe a « mesonic » nucleon field interacting with itself—the linear, local PS-PS theory might perhaps be viewed as a « large-distance-approximation » of the non-linear, non-local PS-PV theory.

A more detailed discussion of the mathematical principles used in the present paper will be presented in a forthcoming paper (to be referred to as (II)); a justification of the methods used below will be given there with such mathematical rigor as should be sufficient for physics. Succeeding papers (quoted as (III) and (IV)) will deal with the potentials derived from PS-PV theory and with renormalization problems, respectively. In the Appendix to this paper a brief exposition of the theorem of König will be given together with a set of formulas used in the text.

2. – Field theories of the 1-st and 2-nd kind.

Following HEISENBERG (3), interactions of quantized fields are divided into two classes, *viz.* the interactions of the 1-st and 2-nd kind. In theories of the 1-st kind (*e.g.*, quantum electrodynamics, PS-PS theory; coupling constants being dimensionless) the coupling strength changes very slowly with energy, the effects of the proper fields of the source particles are small and the transition elements do not increase with increasing energy. In interactions of the second kind (*e.g.*, β couplings, PS-PV theory; coupling constants have the dimension of some positive power of a length) the transition elements increase with increasing energy, the dynamical effects of the proper fields become pre-

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and A. A. LOGUNOV: *Dispersion Relations for Weak Interactions* (preprint). I wish to express my gratitude to Drs. BOGOLJUBOV and ŠIRKOW for communications in advance of publication.

(9) S. HORI and A. WAKASA: *Nuovo Cimento*, **6**, 304 (1957).

dominant, giving automatically strong coupling at high energies. Multiple meson production is favoured in PS-PV theory.

The above classification also divides the interactions into renormalizable and non-renormalizable ones as long as renormalization theory refers to the same type of perturbation theory in both cases ([10]). However, ARNOWITT and DESER ([1]) and COOPER ([2]) have succeeded to show, by using functional integral methods, that a number of interactions of the 2-nd kind actually can be renormalized if one dispenses with the familiar coupling constant expansions. We show in Sect. **4** and **5** that also the customary perturbation theory can be rendered mathematically meaningful in theories of the 2-nd kind (cf., also Ref. ([2])), the non-renormalizable infinities being automatically eliminated. Hence, a classification of relativistic quantized field theories into renormalizable and non-renormalizable ones fails and renormalizability loses its meaning as a selection principle for «real» physical interactions.

There still remains, in the frame of the familiar S-matrix formalism, the division of interactions into those giving rise to a finite set of topologically independent, primitively divergent graphs (1-st kind theories) and those pertaining to an infinite set of such graphs (2-nd kind theories). Correspondingly we have a finite or an infinite set of topologically independent products of field propagators with overlapping light-cone singularities, *i.e.*, with coinciding space-time arguments. Examples are the electron self-energy product $E = \gamma_\mu S_F(x)\gamma_\mu D_F(x)$ in quantum electrodynamics and the multi-meson propagators $[\Delta_F(x)]^n$ $(n = 1, 2, ...)$ in PS-PV theory (cf., below). This classification of field theories according to the topological structure of S-matrix elements remains unaltered even if part of the series sum to closed form expressions.

It is an immediate consequence of the topological structure of 2-nd kind theories that diagrams with an arbitrarily large (infinite) number of meson lines (vertices which are accumulation points of meson lines) must play a central part in those theories ([18]). In the following we concentrate our attention to PS-PV theory. While the «elementary interaction structure» of PS-PS theory corresponds to the exchange of one single meson, *i.e.*, to a Δ_F-propagator, one may expect, in virtue of the above facts, the «elementary interaction structure» of PS-PV theory to be some transcendental function $G(\lambda^2\Delta_F) = M^2 \sum_{n=1}^{\infty} c_n(\lambda^2\Delta_F)^n$ of Δ_F. The modulus $|\lambda|$ of the complex parameter λ, and likewise $1/|M|$, must have the dimension of a length while the c_n are complex dimensionless matrix coefficients. We may put $|M| = g_0/|\lambda|$, $g_0 > 0$ being a dimensionless constant. Since on the one hand Yukawa's potential should result from G for large distances and since Dyson's equivalence theorem between

([10]) S. SAKATA, H. UMEZAWA and S. KAMEFUCHI: *Progr. Theor. Phys.*, **7**, 377 (1952).

PS-PV and PS-PS theories should likewise be valid in the large distance limit, we may identify g_0 with the coupling constant of PS-PS theory (c_1 being a product of γ-matrices). It follows that $|\lambda|$ is proportional to the modulus $|g|$ of the coupling constant $g = \sqrt{4\pi} f/\mu$ of the linear representation of PS-PV interaction (Equ. (1)). Joining the proportionality constant with c_1, we clearly have $g_0 = 2|gM|$, M being now the nucleon mass. From the discussion in Sect. 5 it follows that the physically admissible transcendental momentum space propagator $\mathcal{F}G$ ($\mathcal{F}$ denotes the Fourier transform) must have order of growth (7) ϱ (in $\sqrt{|k_\nu^2|}$) with $0 < \varrho < 1$. Consequently, the asymptotic behaviour of $G(\lambda^2 \Delta_F)$ for $|x_\nu^2| \to 0$ must be of the type $G \approx \exp[A(\lambda^2/x_\nu^2)^\alpha] - 1$ with $0 < \alpha \leqslant 1$. Since, in virtue of the above arguments, the term linear in λ^2 is proportional to Δ_F we have to take $\alpha = 1$. This implies that, at least for small $|x_\nu^2|$, the multiple propagator G must have the form $G(\lambda^2 \Delta_F) = M^2(\exp[g^2 \Gamma \Delta_F] - 1)$, Γ being a matrix operator.

This is lent support by the fact that the Lagrangian of the linear, neutral PS-PV theory, viz.

$$\text{(1)} \qquad \mathcal{L} = ig\bar{\psi}\gamma_5\gamma_\mu(\partial_\mu \Phi)\psi + L_0\,, \qquad L_0 = \bar{\psi}(\gamma_\mu \partial_\mu + m)\psi - \tfrac{1}{2}(\Box + \mu^{*2})\Phi\,,$$

takes after Dyson's wellknown contact transformation the non-linear form

$$\text{(2)} \qquad \begin{cases} \mathcal{L}' = L' + L_0\,, \\ L' = \bar{\psi} m(\exp(2g\gamma_5 \Phi) - 1)\psi\,, \end{cases}$$

m being the bare nucleon mass (11). The equivalence between (1) and (2) makes the distinction between linear and non-linear interactions rather artificial. Starting from (2), we see that the (formal) relation

$$\langle(\exp[i\beta\Phi(x)] \cdot \exp[-i\beta'\Phi(x')])_+\rangle = \exp[-i\beta\beta'(\Delta_F(x - x') - \Delta_F(0))]\,,$$

with $\Delta_F(x - x') = i\langle(\Phi(x)\,\Phi(x'))_+\rangle$, gives rise to the transcendental propagator

$$\text{(3)} \qquad G_0'(\lambda^2 \Delta_F) = M^2(\exp[2g^2 \gamma_5^{(1)} \gamma_5^{(2)} \Delta_F] - 1)$$

in agreement with our previous estimate. The transition from the linear to the non-linear representation of PS-PV theory corresponds to the « gauge transform » $\psi \to \exp[g\gamma_5 \Phi]\psi$ in $\langle(\bar{\psi}(x)\psi(x'))_+\rangle$ In (3) we have denoted by M

(11) The non-linear interaction (L') actually results from the mass term $\bar{\psi} m \psi$ in L_0 by the transformation $\psi \to \exp[g\gamma_5 \Phi]\psi$.

the « renormalized » nucleon mass,

$$M = mZ_2^2 , \tag{4}$$

and

$$Z_2 = \exp\,[-g^2 \Delta_F(0)/2] \tag{5}$$

corresponds to a Green-function renormalization. The propagator (3) can be shown ([1,12]) to represent a first order approximation in M^2 of the two-nucleon propagator and thus forms the analog to the Møller interaction. The parameter g obviously plays not longer the role of a coupling constant in the non-linear theory so that the familiar concept of a coupling between a meson and a nucleon field may no longer prove right. The strength of the nucleon-nucleon interaction is now given by the nucleon mass itself and the parameter g would seem to be responsible for an inner mesonic structure of the nucleons. This means that the separation into a Φ-field coupled to a ψ-field is not reliable. We come back to these questions in Sect. **6.**

By applying Wick's rules to each term of the formal power series expansion of L',

$$L' = \overline{\psi}\,[m \sum_{n=1}^{\infty} (2g\gamma_5\Phi)^n/n!]\psi , \tag{6}$$

the formal S-matrix can be constructed. In particular, $[\Phi(x_1)]^n$ gives rise to a vertex with n meson lines which likewise can be joined with n corresponding lines originating in a vertex at x_2. Thus, multiples $[g^2\gamma_5^{(1)}\gamma_5^{(2)}\Delta_F(x_1 - x_2)]^n$ of Δ_F are generated which, after an appropriate classification of the graphs, can be summed to exponential propagators of the type (3). Joining a vertex with itself produces multiples of $\Delta_F(0)$, so giving rise to Z_2. Those quantities can be eliminated by a modification of Wick's products. Multiple Fermion propagators can be dealt with in a similar manner and vacuum polarization effects may thus be taken into account. If finally the S-matrix expansion in powers of g^2 is regrouped into an expansion in powers of M^2, *i.e.*, into an expansion in terms of the new interaction unit G_c', Equ. (3), one arrives at the series expansions obtained by AD via functional methods (omitting vacuum polarization) ([12]).

The formal perturbation analysis in powers of g^2 presupposes the mathematical existence of the multi-meson propagators $(\Delta_F)^n$ (and that of $(S_F)^n$).

([12]) Since this work has been done, I. S. R. Chisholm (*Phil. Mag.*, **1**, 338 (1956)) has discussed in detail the classification of the graphs resulting from the Lagrangian (6). I am indebted to Prof. A. Salam for calling my attention to Chisholm's work.

Those multiple propagators arise also in a perturbation analysis of Fermi interactions and likewise in the perturbation solution of the basic equations of LSZ for theories of the second kind. The fact that in some cases the series sum to closed form expressions such as (3) and (7) obviously does not resolve the difficulties since those expressions are as meaningless as the multiple propagators themselves. A « definition » of closed form expressions by analytic continuation ([1,2]) goes around the basic problem and merely hides the essential mathematical structure involved in the formalism. In addition, those methods make sense only if closed form solutions actually are available what generally is not the case (example: PS-PV theory). In Sect. **4** we outline a theory of multiple propagators by means of which field theories of the second kind can be treated in terms of standard perturbation analysis.

3. – Point structure and relativistic interaction.

The objects of our attention are the multiple propagators $[\Delta_F(x)]^n$ $(x = x_1 - x_2)$, describing the simultaneous emission, by a point-nucleon at space-time point x_1, of n point-like mesons which are likewise absorbed simultaneously at x_2. x_1 and x_2 are points of the (mathematical!) flat space-time continuum R_4 (Minkowski space). As long as arbitrarily small space-time measurements in R_4 are permitted, the product $[\Delta_F(x)]^n$ is mathematically meaningless within the frame of classical or distribution analysis. Without abandoning the concept of point-particles in R_4 we then must either look for a mathematical formalism which gives rigorous meaning to the products in question or we may not longer insist on such concepts as creation (annihilation) of more than one particle at a single point and subsequent causal propagation. We intend to show in this section that the first alternative entails the second one. This implies that the only available direct relativistic interaction, *i.e.*, the contact interaction which acts if and only if the particles are at the same space-time point, is mathematically inconsistent except some basic (mathematical and physical) axioms are abandoned. This is an immediate consequence of a recent mathematical theorem due to KÖNIG ([5]) (c.f., Appendix) and applies to both field theories of the 1-st and 2-nd kind.

Consider a « physical » point-particle. Its property of « being point-like » is, by definition, expressed in terms of a mathematical object, say, A, associated to the particle, the object A having point-like support ([13]) in the mathematical R_4-continum. In particular, we assume those objects to be elements of a vector space $K = \{A, B, ...\}$ which also contains all continuous functions to account for non-point-like structures. This vector space is assumed to satisfy a set of basic axioms listed in Appendix I. The most important ones are: (i) existence of a continuous infinitesimal displacement operator for all ele-

ments of K; (ii) existence of a product $A \circ B$, valued in K, and subject to a number of postulates (such as $1 \circ (A \circ B) = A \circ B$); (iii) existence of corresponding mappings in R_4 and K which generalize the familiar rules of co-ordinate transformations for functions, and independence of the products of the co-ordinate system. Schwartz's distributions are contained in a subspace K_0 of K via an isomorphism. The support in R_4 of the elements of K is defined algebraically (cf. Ref. ([13])). In particular, Dirac's δ is element of K and satisfies the relation $x \circ \delta = 0$ ([14]). Obviously, δ is an algebraic quantity; no functional concepts being implied. The support of δ (and that of $\delta^{(n)}$) consists of one single point (the origin) of the R_4-continuum.

The geometric point-character in R_4 of a point-particle is defined by the point-like support of the element A associated with the particle and A may thus be identified with δ or any derivative $\delta^{(n)}$. The geometry of interacting point-particles, *i.e.*, the topology of the space of supporting sets, is then « isomorph » to the algebra of the δ-elements contained in K. Hence, the coincidence of the supports of two interacting point-particles in R_4 is algebraically described by the product of two δ-elements, $\delta \circ \delta$ (or $\delta^{(n)} \circ \delta^{(m)}$), this product being valued in K. The problem of the mathematical consistency of the physical concept of direct interaction of point-particles reduces therefore to the algebraic problem of the existence of the square of Dirac's δ. The fields of interacting point-particles are assumed to form a closed system. Consequently, we can test the position of a particle in R_4 only by the point-particles of the system themselves, non-point-like testing-bodies being of course inadmissible within that scheme. The case of free particles is physically uninteresting since in this case each particle possesses its own R_4. Within a closed system of interacting point-particles, the determination of the position in R_4 of a particle—and, therefore, the definition of the position—is effectuated by the interaction process itself. Within the concept of direct contact interaction at least two particles occupy the same point in R_4, *i.e.*, their point-like supports coincide, thus giving rise to δ-products. From the conceptional point of view it makes no difference whether the theory under consideration is of 1-st or 2-nd kind and whether it is renormalized or not, since in any case the basic concept is the point-character of particles and interactions. There are so far no indications that *a priori* renormalized theories (LSZ) contain a « built-in » cut-off which eliminates the point-structures.

The existence of $\delta \circ \delta$ as an element of an abstract space has been proved by KÖNIG with all mathematical rigour ([15]). KÖNIG himself has recently succeeded in formulating the minimum conditions a vector space must satisfy

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([13]) L. SCHWARTZ: *Théorie des distributions*, **1-2** (Paris, 1950-51); I. HALPERIN, *Theory of distributions* (Toronto, 1952); H. KÖNIG: refs. ([5]), ([15]), ([24]), ([31]).

([14]) We use $x \circ \delta = (x \times y \times z \times t) \circ (\delta_x \times \delta_y \times \delta_z \times \delta_t)$, $\times$ means the direct product.

in order to contain $\delta \circ \delta$ as a legitime element ([5]). The respective theorem is stated in Appendix 1: δ (which satisfies $x \circ \delta = 0$ in K) must be zero if $\delta \circ \delta$ (and likewise $\delta(\sigma) \circ (1/\tau)$) is defined in K and if the axioms (i), (ii) and (iii) stated above (and precisized in the Appendix) are to remain satisfied. Of course, $\delta \circ \delta$, $\delta(\sigma) \circ (1/\sigma)$ etc., may exist, and so may exist the multiple propagators $(\Delta_F)^n$, $(S_F)^n$, in a vector space which renounces the simultaneous validity of axioms (i), (ii), (iii). For example, those quantities are elements of König's spaces F and Π ([15]). However, the topological structure of such a space and the corresponding structure of the space of the supporting sets (which defines the physical space-time) can no longer be subject to all familiar postulates.

The physical consequences of König's theorem are the following ones:

1) To each point-particle a δ-element has been associated the support of which describes the localizability of the particle at a point in R_4. Then the theorem implies that the supports of two or more interacting point-particles cannot coincide in R_4, *i.e.*, not more than one particle can occupy the same space-time point, except the axioms (i), (ii), (iii) are violated. If they are to remain valid, then $\delta = 0$ and consequently the point-particles cease to exist because their supporting set becomes now an empty set. The possible violation of the axioms implies, in particular, that the existence of a continuous, linear infinitesimal displacement operator (axiom (i)) is incompatible with the postulated independence of the co-ordinate system (axiom (iii)) and esp. with invariance under similarity transforms (change of scale). This principle of similarity (invariance of the physical laws under a uniform change of the size of all space-time intervals) ensures the space to be a flat one, i.e., a Minkowskian ([16]). Hence, the absence of axiom (iii) indicates that general relativity must play a role in the point-interaction concept. The connection of the similarity principle with the renormalization group is well-known ([17]).

Since the structure (geometry) of the physical space-time is determined (and defined) in terms of the structure of the space of the supporting sets of the particles and, therefore, in terms of the structure of the space K, the theorem of König tells us precisely in what respect our familiar concepts of interacting point-particles must fail.

2) If one considers renormalization as a mapping of bare into physical states, the theorem tells us that this mapping is ill-defined if both the renor-

([15]) H. König: *Math. Ann.*, **128**, 420 (1955).

([16]) A. Alexandrov: *Helv. Phys. Acta. Suppl.*, **4**, 44 (1956).

([17]) N. N. Bogoljubov and D. Širkov: *Nuovo Cimento*, **3**, 845 (1956); V. Z. Blank and D. V. Širkov: *Nucl. Phys.*, **2**, 356 (1957).

malized and the unrenormalized schemes are subject to axioms (i)-(iii). Namely, the unrenormalized scheme contains propagator products such as the electron self-energy product $E = \gamma_\mu S_F(x)\gamma_\mu \circ D_F(x)$, S_F, $D_F \in K$, and, since both schemes are to be imbedded into K, the validity of the axioms entails $\delta = 0$. Hence the renormalization-mapping is a mapping of the interacting system into a null theory since all point-structures cease to exist (empty supports). This supports the conjecture of LANDAU [18], POMERANČUK *et al.* [19]. Of course, the situation changes if part of the axioms (*e.g.*, (iii)) is abandoned. Products like $\delta \circ \delta$, $S_F \circ D_F$, $(\Delta_F)^n$ will then exist and are, for example, elements of König's spaces F and Π. But they will not longer have the same transformation properties as their regular approximations, such as $E_\lambda = \gamma_\mu S_F(x)\gamma_\mu \cdot D_F^1(x)$ (D_F^1 being Feynman's regularized propagator). Already the logarithmic divergence of E_λ, as $\lambda \to \infty$, shows that invariance under similarity transforms and, therefore, homogeneity of space-time [16] will be lost if one renders the product $\gamma_\mu S_F \gamma_\mu \circ D_F$ mathematically meaningful. For example, scale factors generated by similarity transforms in E cannot be taken outside the product [20,29].

This rises of course no objections against *a priori* renormalized formalisms (LSZ). However, if the point-particle concept still is contained in those theories, the previous arguments still apply.

3) Field theories of the second kind refer to multiple propagators whether they are renormalized or not. A simple example is given by the renormalized nucleon propagator of the non-recoil PS-PV theory with (3-dimensional) gradient coupling in the zero meson mass limit. We have ($\tau = t - t'$)

$$S'(\tau) = \langle (\overline{\psi}(t)\psi(t'))_+ \rangle = Z_2 S'_c \tag{7}$$

and the renormalized propagator (closed form!) is given by

$$S'_c = R(\tau) \exp\left[-\alpha^2 \delta'_-(\tau)\right], \tag{8}$$

where $R(\tau) = \theta(\tau) \exp[-iM\tau]$, $Z_2 = \exp[\alpha^2 \delta'_-(0)]$, the renormalized mass being given by $M = m + i\alpha^2 \delta''_-(0)$. This propagator and its Fourier transform (for complex α) has been discussed elsewhere [2]. Of course, S'_c as well as S' must be imbedded into the space K and König's theorem can therefore not

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[18] L. D. LANDAU: in *Niels Bohr and the Development of Physics* (Edited by W. PAULI) (New York, 1955).

[19] I. Y. POMERANČUK, V. V. SUDAKOV and V. A. TER-MARTIROSYAN: *Phys. Rev.*, **103**, 784 (1956).

[20] W. GÜTTINGER: *Nuclear Physics*, in press.

longer be ignored. However, it must be realized that the space K, which is only of finite dimension, must now be enlarged into a space K^* of infinite dimension to account for products with an infinite number of factors. In contrast to this, theories of the first kind require only a finite-dimensional space owing to the finite set of independent products (say, primitively divergent graphs). The important fact is now that within this extended space K^* the transcendental propagators possess non-local structure. Hence, the space K^* can no longer be subject to all the axioms involved in König's original theorem. Consequently, in K^* there exist multiple propagators *a priori*. In this respect even the unrenormalized theories of the second kind possess a much higher degree of mathematical consistency than unrenormalized theories of the first kind.

4. – Multiple propagators.

4·1. – As to a *heuristic approach* to multiple propagators we assume tentatively that multiple creation of particles at x_1, causal propagation of the events and final annihilation at x_2 can only be approached approximatively, the ultimate limit of absolute coincidence of point-like events being excluded. This assumption implies that the multiple propagator $[\Delta_F(x)]^n$ $(x = x_1 - x_2)$ is to be defined by the « product »

$$[\Delta_F(x)]^n = \text{« lim »} \prod_{i=1}^{n} \Delta_F(\sigma - \varepsilon_i)\,, \tag{9}$$

where « lim » means that the parameters ε_i $(\varepsilon_i \neq 0,\ i \geqslant 2;\ \varepsilon_i \neq \varepsilon_k$ for $i \neq k;$ $\varepsilon_1 = 0)$ are to be taken arbitrarily small without reaching zero values (except ε_1) except in those terms in which an existing limit is reached uniformely and is element of K_0. In (9) we have put $\Delta_F = \Delta_F(\sigma)$ in order to emphasize the dependence of Δ_F on $\sigma = r^2 - t^2$. This « incomplete » limit clearly is mathematically as meaningless as the product $\Delta_F \cdot \Delta_F \cdot \cdots \cdot \Delta_F$ were within the frame of classical or distribution analysis. It still gives us an approach to a formula, *viz.* Equ. (12), which will be derived on the basis of the vector spaces K and Π in the following paper. A brief exposition of this subject will be given in the following subsection [21].

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[21] The order of the events in (9) cannot be changed continuously ($|\varepsilon_{i+1}| = \alpha|\varepsilon_i|$, $\alpha \neq 1$). The limit in (9) is a special case of the « limes generalis » used by Methée (ref. [22]); those techniques can be fully justified in terms of a residue-class algebra. The similar lim-process gives zero value to the photon self-energy as has been pointed out by J. Tiomno in conversations.

In this and the following section we confine ourselves to the limit of vanishing meson mass, postponing a discussion of the case $\mu^* \neq 0$ to Sect. **6**. With $\Delta_F \to D_F$ we obtain for an infinitesimal ε, up to terms of order $o(\varepsilon)$, according to (A.8) the relation

$$D_F(\sigma) D_F(\sigma - \varepsilon) = (-1/2\pi^2) D_F'(\sigma - \varepsilon) - (1/4\pi^2)\delta(\sigma)\delta(\sigma - \varepsilon) . \tag{10}$$

The quantity $D_F'(\sigma) = \mathrm{d}D_F/\mathrm{d}\sigma$ is a well-defined distribution. No difficulties arise in (10) if ε is kept arbitrarily small but different from zero. Hence, according to (9),

$$[D_F]^2 \equiv \text{« } \lim_{\varepsilon \to 0} \text{ » } D_F(\sigma) D_F(\sigma - \varepsilon) = (-1/2\pi^2) D_F'(\sigma) , \tag{11}$$

since for $\varepsilon \to 0$, $D_F'(\sigma - \varepsilon) \to D_F'(\sigma)$ uniformely ($D_F' \in K_0$). More generally, for $n \geqslant 1$,

$$[D_F]^n = \text{« } \lim_{\varepsilon_i \to 0} \text{ » } \prod_{i=1}^{n} D_F(\sigma - \varepsilon_i) = (-1/2\pi^2)^{n-1} D_F^{(n-1)}(\sigma)/(n-1)! . \tag{12}$$

The derivatives $D_F^{(\gamma)}(\sigma) = \mathrm{d}^m D_F/\mathrm{d}\sigma^m$, Equ. (A.3), are well-defined distributions which possess finite Fourier transforms (c.f., (A.4,5)); they have been discussed extensively by METHÉE and BRAGA [22,23].

4·2. – Consider now instead of $[D_F]^2$, Equ. (11), the mathematically well-defined product $D_F \circ D_F$. $D_F \circ D_F$ is element of K with at least one axiom missing and thus it may be regarded as element of König's space Π. It can be shown that in Π the relation holds

$$D_F(\sigma) \circ D_F(\sigma) = (-1/2\pi^2) D_F'(\sigma) - (1/4\pi^2)\delta(\sigma) \circ \delta(\sigma) , \tag{13}$$

which also can be inferred from Equ. (10). Hence the quantity (11) makes up part of the rigorous product (13). We have $D_F' \in \Pi_0$ but $D_F \circ D_F \in \Pi$, $\Pi_0 \simeq K_0$ being isomorph to the space of distributions of finite order $\mathcal{D}'^{(m)}$ [13]. This isomorphism is not complete, however, since $\mathcal{D}'^{(m)}$ is subject to axiom (iii) while Π_0 is not, if $\delta(\sigma)$ in (13) is different from the zero element of the algebra.

The space K with axiom (iii) missing is isomorph to Π, Π itself being a

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[22] P. D. METHÉE: *Comm. Math. Helv.*, **28**, 255 (1954); *Compt. Rend. Acad. Sci.*, **240**, 1179 (1955); **241**, 684 (1955); *Int. Colloqu. CNRS, Publ.* (Nancy, 1956), p. 145.

[23] C. L. R. BRAGA: *Lorentz Invariant Distributions* (preprint), to be published. The author is much indebted to Prof. BRAGA for valuable communications prior to publication.

quotient algebra (residue-class algebra). KÖNIG as shown [24] that there exists a continuously infinite set of possible and realizable product spaces Π_ζ; this is a consequence of Zorn's maximal principle. If $D_F \circ D_F$ is constructed in those spaces it follows that the term $(-1/2\pi^2)\, D'_F$ in (13) remains the same for all spaces while $\delta(\sigma)\circ\delta(\sigma)$ takes different values in different spaces Π_ζ. Moreover, there exists always a Π_{ζ_0} in which $D_F \circ D_F$ is given by $(-1/2\pi^2)D'_F$, *i.e.*, by the relation (11). However, a change of the co-ordinate system (*e.g.*, a similarity transform) leads from Π_{ζ_0} to Π_{ζ_1} $(\zeta_1 \neq \zeta_0)$ and this gives an additional contribution to (11). This is due to the absence of axiom (iii) in Π.

4·3. – We need not discuss this dependence of the product space on the co-ordinate system since the same problem is already involved in the quantities $\delta^{(n)}(\sigma)$, $\sigma = r^2 - t^2$, in $D_F^{(n)}(\sigma)$. Namely, those quantities depend on the co-ordinate system too, except for $n = 0$ [20]. This has been shown in detail by BRAGA [23]. The origin of this peculiar behaviour lies in the fact that $\delta^{(n)}(\sigma)$, $n \geqslant 1$, contains pseudofunction concepts in the defining equations. It is well known that for the pseudofunctions the esistence of a continuous linear derivative operation is not compatible with the usual rules of co-ordinate transforms [23,25]. The non-homogeneity of $\delta^{(n)}(\sigma)$, $D_F^{(n)}$ $(n \geqslant 1)$ under a change of the scale must carefully be taken into account whenever the underlying field theory postulates invariance under similarity transforms. This dependence of the products on the co-ordinate unit is the price we have to pay for rendering those quantities mathematically meaningful.

4·4. – Perhaps the most striking property of the product $D_F \circ D_F \circ \cdots \circ D_F$ (and of $D_F^{(m)}$) is that these multiple propagators give rise to repulsive-core potentials. In particular, we have according to (13) $\operatorname{Im}(D_F \circ D_F) = \delta'(\sigma)/4\pi^3$ and by time integration

$$(14a) \qquad \int_0^\infty \mathrm{d}t \operatorname{Im}(D_F \circ D_F) = \frac{1}{4\pi^3}\int_0^\infty \mathrm{d}t\, \delta'(r^2 - t^2) = -\frac{1}{16\pi^3}\operatorname{Pf}\frac{1}{r^3} + \frac{1}{4\pi^2}[1 - \log 2]\,\delta^{(3)}(\boldsymbol{r}) .$$

More generally, for $n \geqslant 1$,

$$(14b) \qquad \int_0^\infty \mathrm{d}t \operatorname{Im}(D_F \circ D_F \circ \cdots \circ D_F)_{(n)} = \frac{\pi}{(-2\pi^2)^n (n-1)!}\int_0^\infty \mathrm{d}t\, \delta^{(n-1)}(r^2 - t^2) = -\frac{\sqrt{\pi}\,\Gamma(n - \frac{1}{2})}{2\Gamma(n)(2\pi^2)^n}\operatorname{Pf}\frac{1}{r^{2n-1}} + \frac{[\log(\gamma) + \psi(n - \frac{1}{2})]}{\Gamma(n)\Gamma(n-1)(8\pi^2)^{n-1}}\,\Delta^{n-2}\,\delta^{(3)}(\boldsymbol{r}) .$$

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[24] H. KÖNIG: *Math. Ann.* (to be published); *Proc. Int. Math. Congr.* (Amsterdam, 1954).

[25] W. GÜTTINGER: *Zeits. f. Naturfor.*, **10***a*, 257 (1955).

Thus, for example, the potential deduced from the propagator $D_F \circ D_F$ gives rise to a repulsive three-dimensional δ-potential beside of the familiar attractive $1/r^3$-potential. Equation (14*b*) follows, for example, from (A.4) with $k_0 = 0$ and application of Schwartz's formula (VII, 7;14) ([13]); Δ^m is the m-fold iterated Laplacian. It should be pointed out that the δ product in (13) does not contribute to the time integral (14*a*).

The δ-cores in (14) are manifestations of the fact that the multiple propagators $D_F \circ D_F$ etc., are not completely causal propagators. That acausal propagators give rise to repulsive potentials is well known and can already be inferred from the change of the sign in the Coulomb potential derived from D_F, if we go over to the acausal propagator by changing the sign of « Feynman's ε » in D_F (A. 3). In (14) we merely have acausality « in the small ». It is worth mentioning that a change of the unit in $D_F \circ D_F$, *i.e.*, in $\delta'(\sigma)$, say, $\sigma \to \lambda^2\sigma$, produces an additional term $\log(\lambda)\cdot\delta^{(3)}(\boldsymbol{r})$ on the right side of (14*a*). This is due to the peculiar behaviour of the pseudofunctions involved in the functional representation of $\delta^{(n)}(\sigma)$ (cf. Ref. ([20]) and ([23])) under similarity transforms. The same result follows from a definition of $\delta^{(n)}(\sigma)$ by analytic continuation methods.

4·5. – It is important to realize that the relation

$$(15) \qquad (D_F(\sigma)\circ D_F(\sigma)\cdot\cdots\circ D_F(\sigma))_{(n)} = (-1/2\pi^2)^{n-1} D_F^{(n-1)}(\sigma)/(n-1)!\,,$$

valid in K with axiom (iii) missing, results almost immediately if the vector space K is subject to the additional postulate that a generalized Fourier operator exists in K, *i.e.*, an operator $\mathcal{F}$ which possesses the property that

$$(ig_{\mu\nu}\partial/\partial k_\nu)\mathcal{F}T = \mathcal{F}(x_\mu \circ T)\,, \qquad (\Box_x)^n \mathcal{F}T = \mathcal{F}(\sigma^n \circ T)$$

for every element $T = A \circ B \circ \cdots \circ C$ of K ($T = \Delta_F \circ \Delta_F$ etc.). This rather general axiom—which characterizes the operator $\mathcal{F}$ as a Fourier operator without referring to particular properties—can hardly be abandoned in field theory since the momentum space propagators (and *not* the co-ordinate space propagators) are the actual physical quantities which are comparable with experimental facts. For example, scattering amplitudes are proportional to the Fourier transforms of the propagators ([2]). The operator $\mathcal{F}$ is an entirely algebraic quantity within K. $\mathcal{F}$ may define an automorphism in K but not necessarily in the extended space K^* which is basic for field theories of the second kind. In particular, in K^* the operator $\mathcal{F}$ will not necessarily have a unique inverse nor will unitarity be strictly preserved. The details of the formalism based on the existence of the operator $\mathcal{F}$ will be discussed in the succeeding paper (II).

We remark finally that the transition from $D_F \circ D_F$ to D'_F means, physically speaking, that the two independent (zero mass) mesons form a structure which may be called a « dipole-particle ». The fact that the potential (14*a*), deduced from $D_F \circ D_F$, contains a repulsive δ-core which compensates the (Pf) $1/r^3$-potential (note that we actually have a pseudofunction which is finite everywhere in contrast to the familiar $1/r^3$-potential) at small distances, implies that the two particles do not arrive strictly simultaneously at the point occupied by the second nucleon.

4·6. – Finally we wish to stress that the non-localizability of field theories with derivative couplings, to be discussed in the following sections, does *not* depend on whether we use the formula (15) or the formula (13) and the corresponding higher order products. It does likewise *not* depend on the restrictions which the theorem of König sets on the unlimited localizability of point-like events in R_4. Instead, the non-localizability of field theories of the second kind is solely due to the fact that the number of topologically independent propagator products in those theories is infinitely large, no matter how these products actually are defined.

5. – Non-local structure of field theories of the second kind.

5·1. – The most striking property of field theories of the 2-nd kind is that they lead to processes with infinite multiplicity already in the lowest order approximation of the theory. The propagators associated with those processes are transcendental functions of Δ_F, S_F etc. These vacuum expectation values are to be considered as elements of the infinite-dimensional vector space K^*, independently of whether the series sum to closed form expressions. In the following we confine ourselves to a discussion of the propagator G'_σ, Equ. (3), of neutral PS-PV theory in the limit of vanishing meson mass μ^*, postponing a discussion of the case $\mu^* \neq 0$ to Sect. 6. Other derivative coupling theories (such as charged PS-PV theory and non-recoil approximations) reveal essentially the similar structures.

G'_σ, with Δ_F replaced by the zero meson mass propagator D_F, is defined in terms of its formal power series expansion, *viz.*

$$G'_\sigma = M^2 \sum_{n=1}^{\infty} [2g^2 \gamma_5^{(1)} \gamma_5^{(2)} D_F(\sigma)]^n / n! \,. \tag{16}$$

In virtue of (15) and (A.3) G'_σ can be written as

$$G'_\sigma = I_\sigma + iH_\sigma \,, \tag{17}$$

where $(\sigma = -x_\mu^2 = r^2 - t^2)$

$$(18) \qquad I_c = M^2 \sum_{n=0}^{\infty} \mathrm{Pf}\,(g^2\gamma_5^{(1)}\gamma_5^{(2)}/\pi^2\sigma)^n/n! = M^2\,\mathrm{Pf}\,(\exp\,[2g^2\gamma_5^{(1)}\gamma_5^{(2)}D^{(1)}(\sigma)] - 1)$$

and

$$(19) \qquad H_c = \pi M^2 \sum_{n=0}^{\infty} \frac{(-\gamma_5^{(1)}\gamma_5^{(2)}/\pi^2)^{n+1}}{n!\,(n+1)!}\delta^{(n)}(\sigma/g^2)\,.$$

Neither I_c nor H_c are distributions but may be considered as boundary values (« Randverteilungen ») of distributions. The Fourier transforms of G_c', I_c and H_c are non-tempered distributions (in K^*).

We wish to stress that Arnowitt and Deser's statement, *viz.* that the δ-term in D_F and Δ_F' does not contribute to the exponentials, is not correct. This may be demonstrated by the following simple counter-example (cf., (II)). According to (A.7) we have

$$\mathcal{F}(\exp\,[-\alpha\delta_-(x)] - 1) = -2\theta(k)\sqrt{\pi\alpha/2k}\,J_1(\sqrt{2\alpha k/\pi})\,,$$

whereas

$$\mathcal{F}(\exp\,[-\alpha/2\pi i x] - 1) = -[\varepsilon(k) + \varepsilon(\alpha)]\sqrt{\pi\alpha/2k}\,J_1(\sqrt{2\alpha k/\pi})\,,$$

and

$$\mathcal{F}\,\{\mathrm{Pf}\,(\exp\,[-\alpha/2\pi i x] - 1) = -\varepsilon(k)\sqrt{\pi\alpha/2k}\,J_1(\sqrt{2\alpha k/\pi})\,.$$

We come back to this in the succeeding papers, in which we also study the properties of I_c, Equ. (18).

The term H_c, Equ. (19), describes the geometric, say, local or global structure of the theory in the same sense as, for example, the δ-function in D_F and S_F describes the local structure of free particle propagators. The series (19), and similarly (16)–(18), cannot be considered as a power series in

$$(20) \qquad \lambda^2 = g^2\gamma_5^{(1)}\gamma_5^{(2)}/\pi^2$$

formed on the coefficient domain $\{\delta^{(n)}(\sigma)\}$ but it may be regarded as an element of an abstract space $\mathcal{H}$ (cf., ([20])). Those infinite series of derivatives of δ-functions have been studied from the algebraic point of view by Schönberg and Braga ([26,27]). In particular, Braga has investigated the topological structure of spaces formed by particular classes of those series. Formally, λ may be taken as a complex variable.

([26]) M. Schönberg: *Quantum Mechanics and Geometry* (preprint); *Ann. Acad. Sci. Bras.* (in press).

([27]) C. L. R. Braga: to be published.

Although, therefore, the series (19) can be treated entirely by algebraic methods—and this is the most general concept—it is still convenient to represent H_c by a functional on a space of indefinitely differentiable functions $\varphi(x_\mu)$. It should be stressed, however, that the series in question are not distributions and that no physical meaning should be attributed to the functions φ [28]. For the sake of simplicity we consider $\sigma = r^2 - t^2$ as one single (real) variable although by this simplification some particular properties of the series are lost, namely those which refer to the transformation properties of the quantities $\delta^{(n)}(\sigma)$ as discussed in Sect. **4.** [20,29]. With the well known formula (it is merely a definition!) $\delta^{(n)}[\varphi] = (-1)^n \varphi^{(n)}(0)$, where now $\varphi = \varphi(\sigma)$, we then have the functional representation of H_c:

$$H_c[\varphi(\sigma)] = -\pi M^2 \sum_{n=0}^{\infty} \frac{(\lambda^2)^{n+1}}{n!\,(n+1)!}\,\varphi^{(n)}(0)\,. \tag{21}$$

We may likewise describe H_c by a functional on a space of analytic functions $\varphi(z)$ $(z = \sigma + i\eta)$. Application of Cauchy's theorem to $\varphi^{(n)}(0)$ in (21) yields then the functional

$$H_c[\varphi(z)] = \frac{1}{2\pi i} \oint_C \mathrm{d}z\, h_c(z)\varphi(z)\,. \tag{22}$$

Here,

$$h_c(z) = -\pi M^2 (\exp[\lambda^2/z] - 1)\,, \tag{23}$$

and $\varphi(z)$ is assumed to be analytic inside and on the circle C of arbitrary radius around $z = 0$ in the complex z-plane.

5·2. – Any infinite series of derivatives of δ-functions

$$H = \sum_{n=0}^{\infty} \frac{(-\alpha^2)^n}{n!}\, c_n \delta^{(n)}\,, \tag{24}$$

may be discussed in terms of those functional representations. In (24) we have put $\delta^{(n)}(\sigma) = \delta^{(n)}$, α is a complex parameter with $|\alpha|$ having the dimen-

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[28] Within a closed system of interacting point-particles only the point-particles themselves can act as test-bodies. Hence, the testing functions $\varphi(x_\mu)$ should *not* be considered as space-time densities of (macroscopic) test-bodies in the sense of Bohr and Rosenfeld. Cf., however, SCHMIDT and BAUMANN, ref. [6].

[29] The non-invariance under similarity transforms (cf., also Appendix (A.4)) in K with axiom (iii) missing (e.g., in Π) brings a new degree of freedom into the theory. The formalism is still Lorentz invariant in the sense that each finite partial series of (19) is so.

sion of a length and the c_n are complex dimensionless coefficients different from zero.

The series (24) possesses the functional representation

$$H[\varphi(\sigma)] = \sum_{n=0}^{\infty} \frac{(\alpha^2)^n}{n!} c_n \varphi^{(n)}(0) , \tag{25}$$

where $\varphi(\sigma)$ is an indefinitely differentiable function of the real variable σ. The complex representation is given by

$$H[\varphi(z)] = \frac{1}{2\pi i} \oint_C \mathrm{d}z\, h(z)\varphi(z) , \tag{26}$$

with $\varphi(z)$ analytic and

$$h(z) = \sum_{n=0}^{\infty} c_n (\alpha^2)^n / z^{n+1} . \tag{27}$$

C is a circle around $z = 0$ in the convergence domain of the series (27) ([30]).

Field theories of the second kind are characterized by transcendental propagators which according to (15) contain infinite series of the type (24). This is an immediate consequence of the postulated existence of a transformation theory in K^*, according to Sect. 4·5. Field theories of the 2-nd kind pertaining to derivative couplings are characterized by functions $h(z)$ which possess an essential singularity at $z = 0$, and in some cases an additional branch point in $z = 0$. They are likewise characterized by the fact that the associated series H possess Fourier transforms (obtained by Fourier analyzing (24) term by term by means of the operator $\mathcal{F}$) which are non-tempered distributions of order of growth ([7]) ϱ with $0 < \varrho < 1$ (cf., (II)). We shall show in the following that those series represent non-local quantities which impart to field theories with derivative couplings a non-local, say, global structure.

Consider first the finite series

$$H_N = \sum_{n=0}^{N} \frac{(-\alpha^2)^n}{n!} c_n \delta^{(n)} , \tag{28}$$

with the functional representation

$$H_N[\varphi(\sigma)] = \sum_{n=0}^{N} \frac{(\alpha^2)^n}{n!} c_n \varphi^{(n)}(0) , \tag{29}$$

([30]) Since this work has been done, the paper of Gel'fand and Silov (*Am. Math. Soc. Trans.*, **5**, 221 (1957)) became available. These authors have treated the case $\varrho \geqslant 1$ by similar methods. However, the case $0 < \varrho < 1$ has not been discussed. I am grateful to Prof. B. Gross for calling my attention to Gel'fand and Silov's paper.

N being an arbitrarily large but finite integer. The distribution H_N has the point $\sigma = 0$ (light-cone) as support for any finite N. This support is independent of the size of α and the value of $H_N[\varphi(\sigma)]$ depends only on the values of $\varphi(\sigma)$ and its first N derivatives on that support. Those finite series are typical of field theories of the first kind and the set of numbers $\varphi^{(n)}(0)$ may be interpreted as representing a finite set of position measurements necessary to determine the local structure of H_N.

On the other hand, the value of the functional $H[\varphi(\sigma)]$, (25), and so that of $H_c[\varphi]$ (19), depends on the values of $\varphi(\sigma)$ and *all* its derivatives at the point (light-cone) $\sigma = 0$. Consequently, H (and so H_c) depends also on the values of φ in a certain real, open neighbourhood of $\sigma = 0$. That is, the value of $H[\varphi]$ depends on and is determined by the values of φ and all its derivatives at a point (hyperboloid) $\sigma_0 \neq 0$, or what is the same, it depends on the values of $\varphi(\sigma)$ on an infinite, bounded set of points (hyperboloids) $\sigma_i \neq 0$, having $\sigma = 0$ or any other point $\sigma' \neq 0$ as accumulation point. Consequently, H, and likewise H_c, is *not* localized at the origin since, if it would possess only the origin $\sigma = 0$ as support, it could not depend on the values of $\varphi(\sigma)$ at points different from the origin. Therefore, H is a non-local, *i.e.*, a global structured quantity.

There are different classes of series H which possess different types of non-local structures. In particular, there are cases in which H has a supporting set consisting of a finite number of isolated points. Those series, $H = H_0$, are characterized by the fact that their Fourier transforms possess order of growth $\varrho = 1$. That is, the functions $h_0(z)$ associated with H_0 are meromorph and possess a finite set of isolated poles within a circle of finite radius around $z = 0$. For example, the series

$$H_0 = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \delta^{(n)}, \tag{30}$$

with $\delta = \delta(x)$ (one-dimensional case), represents the Fourier transform of $\exp[\alpha k]$ and gives rise to $h_0(z) = 1/(z + i\alpha)$. The circle C around $z = 0$ has radius $|z| = |\alpha| + \varepsilon$ $(\varepsilon > 0)$. $|\alpha|$ determines a « critical length » and from (26), by continuing $h_0(z)$ into the circle C, we have the functional $H_0[\varphi(z)] = \varphi(-i\alpha) = = « \delta(x + i\alpha) »[\varphi(z)]$. In this case $H_0[\varphi(z)]$ depends on the value of $\varphi(z)$ at the point $z = -i\alpha$, but, since $\varphi(z)$ must be analytic, we also must consider a neighbourhood of that point. However, we still can define the support of $H_0[\varphi]$ to be the single point $z = -i\alpha$ since we can test this support by means of an analytic function $\varphi = \chi(z)$ which vanishes precisely (and solely) at this points. If α is not purely imaginary we must of course use analytic functions.

It must be realized, however, that the non-local structure of H_0—we have a dislocation of the support from $z = 0$ to $z = -i\alpha$—is *not* due to the fact

that we have used analytic functions which of course would automatically bring global structures into the formalism. Namely, we may likewise use the functional representation (25) in which no use is made of analytic functions to describe, for instance, the case $\alpha = ix_0$ (x_0 real). This series has the point $x = x_0$ as support and is « isomorph » to $\delta(x - x_0)$ in the sense that Schwartz's space of δ-distributions is isomorph to a subspace $\mathcal{H}_0$ of the space $\mathcal{H}$ of the series (24).

We wish to stress that the non-local structure of the series H, H_c also is *not* a consequence of the fact that functional representations with indefinitely differentiable functions $\varphi = \varphi(\sigma)$ (σ real) have been used. In fact, those series are formal series on the field $\{c_n \delta^{(n)}\}$ and form a modul if addition is defined term-wise. Thus they are purely algebraic elements of an abstract space $\mathcal{H}$ and, as has been shown by BRAGA ([27]), their local and global structures can be discussed algebraically without any reference to functional representations. Thus, for example, we do not need analytic functions in the discussion of H_0 even if α in (30) is not purely imaginary, *i.e.*, even if the support of H_0 lies in the complex plane. That the support concept can be defined entirely in terms of algebraic concepts has been shown by KÖNIG in his algebraic (and axiomatic) approach to distributions and the same concepts also apply to the series H ([31]).

We remark that for $\varrho > 1$ the series H converges only for $|z| \to \infty$ and in this case the representation (25) must be used to show that the support of the series with $\varrho > 1$ covers a non-point-like region of the z-plane which includes the point ∞. Those series behave asymptotically as $\exp[|\sigma/x^2|^\varrho]$ ($\varrho > 1$) for $|\sigma| \to \infty$ and, therefore, they seem not to be of physical interest. The series with $\varrho = 1$, in particular (30), *i.e.*, quantities like $\delta(x+iy)$, $\delta(x^2+y^2)$ are important in dispersion relation theory ([32]). Since those quantities actually are non-local, referring to « neighbourhoods » and not solely to points, und thus must be imbedded into the space K^*, one may conclude that the global structure so encountered with in the « unphysical » regions must necessarily also influence the local structures in the physical regions.

5·3. – Field theories of the 2-nd kind with derivative couplings belong to the class with $0 < \varrho < 1$. The corresponding series H, in particular, the series H_c, (19), imply a kind of non-localizability which is fundamentally different from that encountered with in the case $\varrho \geqslant 1$. Namely, as we will see in a moment, the support of the series with $0 < \varrho < 1$ really is *not localizable*

([31]) H. KÖNIG: *Math. Nachr.*, **9**, 129 (1953).

([32]) K. SYMANZIK: *Phys. Rev.*, **105**, 743 (1957); N. N. BOGOLJUBOV, B. V. MEDVEDEV, M. K. POLIVANOV and D. W. ŠIRKOV: *Problems of Dispersion Relation Theory* (preprint).

in the proper sense of the word, that is, the support is « statistically » indeterminate, in contrast with the well-determined support of H_0, for example.

While in the case $\varrho = 1$ the support actually can be tested and finally consists of a finite, discrete and therefore closed set of points (hyperboloids), this is not longer true in the case $0 < \varrho < 1$. Namely, in this case the function $h(z)$ (cf., for instance, Equ. (23)) has an essential singularity at $z = 0$ and, therefore, in order to test the support of $H[\varphi(z)]$ we would need an analytic function $\varphi = \chi(z)$ which vanishes together with all its derivatives $\chi^{(n)}(z)$ $(n = 1, 2, ...)$ at $z = 0$. Hence it vanishes identically inside and on the circle C and so also vanishes in some region that contains C. It is therefore impossible to fix the support of $H[\varphi]$ $(0 < \varrho < 1)$ and that of H_c inside C and the more accurate measurements are made to determine the support (using $\varphi = \chi(z)$ with $\chi^{(n)}(0) = 0$ for $n \leqslant n_0$ and letting n_0 tend to infinity), the higher becomes the uncertainty of the support.

It is clear that the case $0 < \varrho < 1$ is just that one to which our arguments of the preceding subsection apply in full: $H[\varphi]$ depends on and is likewise determined by the values of $\varphi(\sigma)$ and all its derivatives at $\sigma = 0$. So it depends on the values of φ and all its derivatives at any point $\sigma_0 \neq 0$ and is likewise determined by the values of φ on any given, infinite and bounded set of points (hyperbolois) $\sigma_i \neq 0$ which may have any point $\sigma' \neq 0$ (or even $\sigma = 0$)as accumulation point. Otherwise $\varphi(\sigma)$ would not be indefinitely differentiable or $\varphi(z)$ would not be analytic in contrast to the supposition. Finally, the same result follows if one dispenses with functional representations, as has been indicated above.

Since H_c depends also necessarily on the values of $\varphi(\sigma)$ at points (hyperboloids) σ_i different from the origin (light-cone) $\sigma = 0$, it is obviously a non-local quantity. If it were localized at the origin and so had the point $\sigma = 0$ as support, it would be independent of the behaviour of φ at points different from the origin. In the (rather restrictive) complex functional representation (22) the circle C may have arbitrary radius and this would seem to tell us that the domain of non-localizability may even consist of the point $\sigma = 0$ *plus* a « small » open neighbourhood. However, quite apart from the fact that the representation (21) does not refer to any circle and that even general algebraic methods apply to (19), we still may take the circle C arbitrarily large (of course finite). Thus, for example, $H_c[\varphi(\sigma)]$ depends on and is determined by the values of φ on an infinite set of points σ_i in the interval $a < |\sigma_i| < b$, $0 < a < b < \infty$, a and b being arbitrary numbers.

5·4. – The preceding analysis tells us that H_c is a non-local quantity of a rather unfamiliar structure. H_c plays that role in the propagator G'_c which is played by the δ-function $\delta(\sigma)$ in the elementary propagators Δ_F, S_F. It is worth mentioning that the relation (17) holds formally if the Fourier ope-

rator $\mathcal{F}$, defined in Sect. 4·5, is applied to both sides; thus, in proving the non-localizability one needs not even an explicit construction of the products! In the same sense as $\delta(\sigma)$ in the elementary propagators describes the localizability of the associated point-particles in R_4, the quantity H_c describes the non-localizability of the point-particles in R_4 if these particles are subject to interactions of the second kind. Since the support of H_c consists of an infinite, bounded set of points (or even of a continuum), it follows that (within the familiar meson nucleon interaction concept): (i) the infinite set of point-like mesons involved in G_c' cannot occupy one single point in R_4, *i.e.*, creation (annihilation) of an infinite number of particles cannot be forced to take place at a single space-time point but requires an infinite set of points, that is, a non-point-like space-time domain, and (ii) the positions of two interacting point-like nucleons spread likewise over a non-point-like space-time region.

The support of H_c still has no extention in the sense that a definite circle could be fixed outside of which H_c vanishes identically. Hence, also no structure function in the usual sense is involved in the theory and, in contrast to the general belief ([10]), there is no dependence on the details of the structure of the elementary particles in theories with derivative couplings.

A precise characterization of the spread in R_4 of the support of H_c (which obviously brings a new degree of freedom into the system) would seem, in virtue of the above results, to be merely a question of an appropriate mathematical definition which is to be judged by the physical usefulness of the results. The discussion in the following section indicates that the support of H_c (and that of H with $0 < \varrho < 1$) may properly be described in terms of statistical concepts. We therefore define the « extention » of the support of H_c in terms of an « effective size » or « range » by using the special testing functions $\varphi(x_\mu) = \exp[-k_\mu x_\mu]$ which just yield the formal Fourier transform of H_c: $(2\pi)^4 \mathcal{F} H_c = H_c(\exp[-ik_\mu x_\mu])$. Other particular testing functions may be of use too. $\mathcal{F}H_c$ follows immediately by applying the operator $\mathcal{F}$ term by term to the series (19) and using formula (A.4). Then, in virtue of the classical theorems ([7]) of the estimation of the growth of a function from the coefficients of its Taylor expansion, it follows that $\mathcal{F}H_c$ behaves asymptotically as ($\beta =$ *complex* constant with $|\beta| \simeq 1$)

$$\mathcal{F}H_c \approx \log(g^2 k_\nu^2) \exp\left[\beta (g^2 k_\nu^2)^{\frac{1}{3}}\right], \tag{31}$$

for $|k_\nu^2| \to \infty$. The closed form expression resulting for the non-recoil approximation (7), (8) shows a similar behaviour, as has been discussed in a preceding paper (cf., also (II)).

It is now seen that an alteration of the argument of λ^2, (20), *i.e.*, of g^2 or of k_ν^2, produces an essential change in the behaviour of $\mathcal{F}H_c$ only in the region $|k_\nu^2| \gtrsim |1/g^2|$. A separation into cos- and sin-parts produces multiplicative factors 2 and $2\sqrt{8}$ resp. Note that Ref. ([7]) gives only an estimation

of the absolute value and that actually the exponential contains a complex coefficient (β) with modulus of order one. The region $|k_\nu^2| \gtrsim |1/g^2|$ is that one where $\mathcal{F}H_c$ oscillates (even for real g). On the contrary, the behaviour of $\mathcal{F}H_c$ in the region $|k_\nu^2| \lesssim |1/g^2|$ is essentially independent of the argument of λ^2 (and so of that of g^2, k_ν^2). More generally, the momentum space propagator $\mathcal{F}G_c'$, $\mathcal{F}H_c$ and likewise the vacuum expectation value $\mathcal{F}\langle\overline{\psi}(x)\psi(x')\rangle$ begin to oscillate (also for real g) at $|k_\nu^2| \approx |1/g^2|$ (more precisely at $|k_\nu^2| = a|1/g^2|$ with a of order 1) and oscillate with increasing frequency as $|k_\nu^2| \to \infty$ (the amplitude increases or decreases according to the argument of λ^2), whilst for $|k_\nu^2| \lesssim |1/g^2|$ there are no zeros and in this region the functions are slowly in- or decreasing and their behaviour is essentially independent of the argument of λ^2 ([40]). Cf., also the one-dimensional analog to H_c, $\mathcal{F}H_c$, *i.e.* the quantities $H_1^{(0)}$ and $\mathcal{F}H_1^{(0)}$ in Sect. 6·2.

We therefore define the region $\sqrt{|k_\nu^2|} \gtrsim |1/g|$ in momentum space and similarly the region $\sqrt{|x_\nu^2|} \lesssim g|$ (or even $\sqrt{|x_\nu^2|} \lesssim g/\pi|$ or $\sqrt{|x_\nu^2|} \lesssim |g/2\pi|$) in co-ordinate space as the « effective size » of the support of H_c. In co-ordinate space the domain $|x_\nu^2| \lesssim g^2|$ is essentially that of the oscillations of $h_c(z)$, Equ. (23), and $|x_\nu^2| \approx |g^2|$ determines the distance at which the essential singularity (*i.e.*, the oscillations of $h_c(z)$) « begins ». The physical meaning of this effective size of the support of H_c will be discussed in the following section. It is clear that the length $|g|$, *i.e.*, the parameter λ, plays a role in the structure of the support of H_c which is quite different from that played by the length $|\alpha|$ in the series (24) with $\varrho \geqslant 1$, *e.g.*, in (30), and in the finite series (28). While in (28) the length $|\alpha|$ and its powers describe the weight of the contribution to H_N of the δ terms, the support of H_N is independent of α. On the other hand the support of (24) with $0 < \varrho < 1$ or $\varrho \geqslant 1$ depends strongly on the magnitude of α (cf., Equ. (30)).

We remark that $|g| \approx 1/\mu$ if the familiar value ([33]) $f^2 \approx 0.08$ is used in the relation $g = \sqrt{4\pi} f/\mu$ (μ = meson mass). Thus the effective size of the support of H_c is of the order of the meson Compton wave length. This is not unreasonable since, as we will see in the following, the meson mass parameter (μ^*) involved in the Δ_F-propagator, plays another role in PS-PV theory than in PS-PS theory. However, at the present stage of the theory no numerical predictions are intended.

6. – Discussion.

6·1. – The effective range of the support of H_c determines that region where Dyson's equivalence theorem becomes invalid: $|g| = |g_0/2M|$ holds only

([33]) G. C. Wick: *Rev. Mod. Phys.*, **27**, 339 (1955).

for $|x_\nu^2| \gtrsim |g^2|$ (or $|\boldsymbol{r}| \gtrsim |g|$); in this region only the one-particle propagator Δ_F in G_c' is important (cf. also Equ. (44) of AD).

According to Equ. (14*b*) the « potential » derived from Im G_c', *i.e.*, from H_c, by time integration is given by ($r = |\boldsymbol{r}|$)

$$V(g, \boldsymbol{r}) = \int_0^\infty \mathrm{d}t\, H_c = -\frac{g^2\gamma_5^{(1)}\gamma_5^{(2)}}{2\pi r}\frac{M^2}{\pi}\,\mathrm{Pf}\,{}_1F_1\left(\frac{1}{2};\, 2;\, \frac{g^2\gamma_5^{(1)}\gamma_5^{(2)}}{\pi^2 r^2}\right) + R(g, \boldsymbol{r})\,, \tag{32}$$

where

$$R(g, \boldsymbol{r}) = 2g^2\gamma_5^{(1)}\gamma_5^{(2)}M^2 \sum_{n=1}^{\infty}\left(\frac{g^2\gamma_5^{(1)}\gamma_5^{(2)}}{4\pi^2}\right)^n \frac{[\log\gamma + \psi(n+\frac{1}{2})]}{(n-1)!\,n!\,(n+1)!}\,\Delta^{n-1}\delta^{(3)}(\boldsymbol{r})\,. \tag{33}$$

If g is real and $\gamma_5^{(1)}\gamma_5^{(2)} = +1$, the confluent hypergeometric function ${}_1F_1(\frac{1}{2}, 2; \lambda^2/r^2)$ describes the attractive part of $V(g, \boldsymbol{r})$. The series (33) obviously represents a repulsive-core potential, the support of $R(g, \boldsymbol{r})$ having an effective range of order $|\boldsymbol{r}| \approx |g|$ (similarly to $\mathcal{F}H_c$, $\mathcal{F}R$ has order of growth $\varrho = \frac{2}{3}$). In consequence of the series (33) the potential cannot be called an adiabatic one but requires a statistical interpretation similar to that of H_c. For $r \to 0$, $(1/r)\,{}_1F_1(\frac{1}{2}, 2; \lambda^2/r^2)$ diverges (34) as $+r^2 \exp[g^2/\pi^2 r^2]$ if $\gamma_5^{(1)}\gamma_5^{(2)} = +1$ and tends to the finite value $-\sqrt{4\pi/g^2}$ if $\gamma_5^{(1)}\gamma_5^{(2)} = -1$. For large distances, $V(g, \boldsymbol{r})$ behaves as ($r \to \infty$)

$$V(g, \boldsymbol{r}) \approx -\left[\frac{g^2\gamma_5^{(1)}\gamma_5^{(2)}M^2}{2\pi r} + \frac{g^4M^2}{8\pi^2 r^3}\right]$$

and this holds for $r > r_0 \approx |g/\pi|$, independently of whether g is real or complex. If g is complex, ${}_1F_1(\frac{1}{2}, 2; \lambda^2/r^2)$ begins to oscillate at $r \approx r_0$. Thus $r \approx |g|$ determines the distance where the familiar $1/r^3$-potential becomes invalid. We note that the closed form representation of $R(g, \boldsymbol{r})$ involves a logarithmic factor which changes its sign at $r \approx r_0$. The potential (32) will be discussed in detail in (III), where we will show that the repulsive forces originating from $R(g, \boldsymbol{r})$ begin to act at a distance of the order $|g|$.

The oscillatory behaviour of $\mathcal{F}H_c$, and likewise that of $\mathcal{F}G_c'$ and of $\mathcal{F}\langle\overline{\psi}(x)\psi(x')\rangle$, discussed in Sect. **4**, deserves further investigation. It is an immediate consequence of those oscillations that Wightman's positive-definiteness condition (6) is violated since the suppositions of the generalized Bochner theorem (35) (positive, slowly increasing Fourier transforms) are no longer satisfied. This seems to be a general property of field theories of the second

(34) Note that (32) contains the pseudofunction $\mathrm{Pf}[{}_1F_1(\frac{1}{2}, 2; \lambda^2/r^2)]$ and, therefore, is finite everywhere. Thus, $V(g, \boldsymbol{r})$ is still finite in the neighbourhood of $r = 0$ (cf., III).

(35) L. Schwartz: ref. (13), **2**, 132.

kind. Since, however, these oscillations are restricted to the high energy region $|k_\nu^2| \gtrsim |1/g^2|$ we still are able to establish positive-definiteness conditions which apply to regions $|k_\nu^2| \lesssim |1/g^2|$ and to co-ordinate space regions $|x_\nu^2| \gtrsim |g^2|$. This lack of positive-definiteness clearly corresponds to a lack of microscopic causality within the region $|x_\nu^2| \lesssim |g^2|$; causality may be still preserved « in the large ». The lack of causality is of course related to the repulsive-core potential (33).

These results may be sufficient to display the fundamental role which $|g|$ plays as a « critical length » in PS-PV theory.

6.2. – The support of H_c, Equ. (19), remains essentially unaltered if we replace the propagator D_F in (16) by the Δ_F-propagator (which actually stands in (3)). The mass parameter μ^* involved in $\Delta_F(x;\mu^*)$ and in Φ obviously plays no role at small distances and its influence on the size of the support of the H-series associated with G_c', Equ. (3), is rather secondary. Let us consider, for a small μ^*, the influence of the logarithmic term in $\Delta_F(x;\mu^*)$ on the exponential (16). The effect of this term is to replace (16) by $(x_\nu^2)^{g^2\gamma_5^{(1)}\gamma_5^{(2)}\mu^{*2}\,4\pi^2} \exp[2g^2\gamma_5^{(1)}\gamma_5^{(2)}D_F]$. It is sufficient to consider the one-dimensional analog to $(x_\nu^2)^m \exp[\alpha^2 D_F]$, $m \geqslant 0$, and to $(D_F)^m \exp[\alpha^2 D_F]$, $m \geqslant 1$, *i.e.*, the propagators $G_1^{(m)} = x^m(\exp[-\alpha\delta_-(x)] - 1)$ and $G_2^{(m)} = [\delta_-(x)]^m \exp[-\alpha\delta_-(x)]$ respectively. By the relation (A.7)—the analog to (15)—we find the H-series associated with $G_\mu^{(m)}$, $\mu = 1, 2$, $G_\mu^{(m)} = I_\mu^{(m)} + iH_\mu^{(m)}$, to be (cf., (II)):

$$H_1^{(m)} = \pi(i\alpha/2\pi)^{m+1} \sum_{n=0}^{\infty} (-i\alpha/2\pi)^n \delta^{(n)}(x)/n!\,(n+m+1)!$$

and

$$H_2^{(m)} = -\pi(i/2\pi)^m \sum_{n=0}^{\infty} (-i\alpha/2\pi)^n \delta^{(n+m-1)}(x)/n!\,(n+m-1)!$$

respectively. The Fourier transforms are

$$\mathcal{F}H_1^{(m)} = (1/2)[i\sqrt{\alpha/2k\pi}]^{m+1} I_{m+1}(\sqrt{2\alpha k/\pi})$$

and

$$\mathcal{F}H_2^{(m)} = (1/4\pi i)[\sqrt{k/2\alpha\pi}]^{m-1} I_{m-1}(\sqrt{2\alpha k/\pi})$$

respectively. The range of the support may again be defined, for example, by the first zero of the Bessel functions I_n. It is then seen that in co-ordinate space the size of the support of, *e.g.*, $H_1^{(1)}$ is approximately two third of that of $H_1^{(0)}$.

This exemplifies that the effect of μ^* is merely to modify slightly the inner structure of the support of H_c, the range of the support of the series H asso-

ciated with G'_c in the case $\mu^* \neq 0$ being essentially the same as that of the series H_c corresponding to the case $\mu^* = 0$. Only the length $|g|$ is responsible for the spread of the support of H_c.

6·3. – While g loses completely its meaning as a coupling in the non-linear formalism, the strength of the nucleon-nucleon interaction is now given by the (renormalized) nucleon mass M itself. Hence, the nucleon mass plays here a more fundamental and a more physical role than, for instance, in PS-PS theory in which M is merely a parameter of the same « weight » as the meson mass μ^*. Apparently M would seem to be zero if the value $\sigma = 0$ in the leading term $1/\sigma$ of Δ_F in (4), (5) is approached in a space-like fashion to preserve formally the canonical commutation rules. $M = 0$ would imply that there is no interaction at all! However, this does by no means justify Pomerančuk's results ([13])· Quite apart from the fact that Z_2 as it stands in (5) is meaningless and that the infinite oscillations in the neighbourhood of the light-cone produced by the δ-term in Δ_F cannot be neglected, those formal conclusions are mathematically inadmissible, last but not least, because the theory actually is non-local, although represented in terms of localizable structures.

Actually it is impossible to insist on the preservation of the canonical commutation rules since those rules make sense, in the present formalism, only in the « large distance region ». These problems will be discussed in detail in the fourth paper of this series. Our discussion of the renormalization problem presented there will be based on the equation

$$\exp\left[-g^2 D_F(\boldsymbol{x})/2\right] \circ T_R(\boldsymbol{x}) = \delta^{(3)}(\boldsymbol{x}), \tag{34}$$

which is to be understood as an inversion of the familiar Lehmann-Wightman relation (such as $\Delta' = \int_0^\infty \mathrm{d}\varkappa^2 \varrho(\varkappa^2) \Delta(x; \varkappa^2)$) ([36]). In (34) we have put $D_F(\boldsymbol{x}) = D_F(\boldsymbol{x}, t=0)$, $T_R(\boldsymbol{x})\gamma_4^{\nu\beta} = [\bar{\psi}_R^\varkappa(\boldsymbol{x}_1), \psi_R^\beta(\boldsymbol{x}_2)]_+$, T_R being the renormalized anticommutator for equal times while $\delta^{(3)}(\boldsymbol{x})$ on the right side of (34) is the unrenormalized anticommutator. Equ. (34) must be considered as an equation for the determination of the renormalized anticommutator T_R. Formally, by « division » we get immediately the relation $T_R(\boldsymbol{x}) = Z_2^{-1}\delta^{(3)}(\boldsymbol{x})$ with Z_2 given by (5). However, such a formal « solution » of (34) is as meaningless as the formal solution $T = (x^{-1})\,\delta(x) = (1/0)\,\delta(x) = \infty\delta(x)$ of the equation $x \circ T = \delta$, the exact solution of which is given by $T = -\delta'(x)$ (plus $c\delta$, c arbitrary constant). In reality, as we shall show in (IV), the correct solution of (34)

([36]) H. Lehmann: *Nuovo Cimento*, **11**, 342 (1954); A. S. Wightman, ref. ([6]).

is an infinite series of derivatives of δ of the type (24) with $0 < \varrho < 1$. Hence, T_R is a non-local quantity of similar structure as H_ϱ and the support of the series which represents T_R spreads also over an interval $|x^2| \lesssim |g^2|$. In the case $t \neq 0$ the size of the support of T_R is again of the order $|g|$.

6·4. – The non-localizability of H_C, T_R in space-time continuum R_4 implies an intrinsic indeterminacy in space-time of the light-cone, this indeterminacy being governed by the «fundamental length» $|g|$ which, if our previous estimate is reliable, is of the order of the meson—or even the nucleon—Compton wave length. Owing to the absence of a structure function this indeterminacy of the light-cone in R_4 requires a statistical interpretation: We still may say that the two interacting nucleons are represented by points in the space-time continuum R_4, but the spread in R_4 of the supports of H_C, T_R imparts an intrinsic indeterminacy to the positions x_1 and x_2 of these point-nucleons in R_4. The space-time region, in which the localization of both interacting particles most probably takes place, is given by $|x_r^2| \lesssim |g^2|$ and the interaction spreads in this sense over that space-time volume. In this sense the nucleons possess spatial and temporal «extention». The function $h_C(z)$, Equ. (23), may thus be regarded as a probability-distribution function. This interpretation of the spread of the supports of H_C, T_R obviously accounts for the lack of positive-definiteness in Hilbert space and the lack of causality in the domain $|k_r^2| \gtrsim |1/g^2|$, $|x_r^2| \lesssim |g^2|$ caused by the oscillations of the momentum space operators in those regions.

A statistical interpretation of the non-localizability also is lent support by the fact that the arbitrarily large number of meson lines which causes the non-localizability must, according to LANDAU [18], lead to a relativistic hydrodynamic picture which, for example, describes multiple meson production in terms of a quantum field statistics [37].

It is worth mentioning that the situation encountered with in the present formalism has much in common with that found by PAIS and UHLENBECK in the so-called exponential-modifications of quantized field theories [38] and with Heisenberg's non-linear spinor theory [39]. In particular, the model discussed by PAIS and UHLENBECK involves an infinite series of δ-functions which causes the acausal structure of the theory. It seems likely that those acausal effects are always related to infinite oscillations of the operators in the neighbourhood of the light-cone, and to the non-tempered structure of the associated momentum space operators (propagators and spectral functions). Incidentally it may be observed that the support of H_C, T_R is non-point-like

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[37] H. EZAWA, Y. TOMOZAWA and H. UMEZAWA: *Nuovo Cimento*, **5**, 810 (1957).
[38] A. PAIS and G. E. UHLENBECK: *Phys. Rev.*, **79**, 145 (1950).
[39] W. HEISENBERG: *Rev. Mod. Phys.*, **29**, 269 (1957).

even if $|g|$ is arbitrarily small whereas for $g = 0$ the support of H_c is empty. Since, therefore, the limit $g \to 0$ is highly non-uniform, adiabatic switching on or off the interaction has no meaning at all. The unrenormalized anticommutator $\delta^{(3)}(\boldsymbol{x})$ likewise is reached in an extremely non-uniform manner if $g \to 0$ in T_R.

6·5. – The spread in space-time of the supports of H_c, T_R implies that the infinite set of direct relativistic (contact) interactions, represented by the series (6), which characterizes PS-PV theory, is equivalent to an indirect interaction which, however, is to be interpreted in terms of statistical concepts. The « range » of this interaction is given by the length $|g|$ which is of the order of the meson Compton wave length (Sect. 5·4). On the other hand, we have seen that the mass parameter μ^*, involved in Φ (Equ. (1)) and in the Δ_F-propagator in G'_c (Equ. (3)), looses its meaning as a meson mass in the region $|x_\nu^2| \lesssim |g^2|$ and has no essential influence on the size of the supports of H_c and T_R. This suggests that the non-linear, non-local PS-PV theory gives rise to a nucleon-nucleon interaction in which the non-contact forces, and therefore the range of the interaction, are attributed to and caused by the statistical spread in space-time of the interacting point-nucleons, governed by the fundamental length $|g|$ which now plays the role of the meson Compton wave length. Since, however, the generalized equivalence theorem of Arnowitt and Deser [1] establishes a more closed connection between PS-PV and PS-PS theory than Dyson's theorem, the local, linear PS-PS theory might ultimately be viewed as an approximation for regions $|x_\nu^2| \gtrsim |g^2|$ of the non-local, non-linear PS-PV theory.

It is still an open question whether and how the length g actually should be renormalized, and whether in the final version of the theory g remains real [40]. In view of the new physical meaning of g, which is now no longer a coupling constant as it seemed to be in the linear representation, analogies with familiar renormalization concepts do not seem of much aid. Since, in any case, the theory requires an interpretation which is different from the familiar concept of meson-nucleon interaction, at least in the small space-time regions, the length g may rather be considered as a structure constant not subject to renormalization concepts.

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[40] An inversion of the sign of g in Eqs. (1), (2), say, $g \to g \exp[in\pi]$, produces apparently no change in G'_c, H_c, T_R but nevertheless gives rise to additional contributions to the momentum space operators $\mathcal{F}G'_c$, $\mathcal{F}H_c$ and $\mathcal{F}T_R$ owing to the logarithmic factor (cf. (31)) $\log(g^2k_\nu^2)$ involved in those quantities according to (A.4). These logarithmic terms are the origin of the non-invariance of the theory under similarity transforms (dependence on the coordinate unit). The fact that a change of the sign of g leads in momentum space to another Riemann surface may perhaps be related to an assymetry of G'_c, H_c T_R under space-time reflections.

The above results suggest to consider the non-linear PS-PV interaction as being more fundamental than the linear form, even though the former has been obtained from the latter by a contact transform. Namely, the term $\exp[g\gamma_5\Phi]$ in $\exp[g\gamma_5\Phi]\psi$ (Sect. 2) which produces the exponential propagators and thus causes the non-localizability, might perhaps be viewed as a statistical weight function ($\sim h_c(z)$) which attributes an intrinsic non-localizability to the point-nucleons in space-time. Moreover, since the effect of this term is rather different from that of any familiar meson-nucleon interaction—although it represents an infinite set of mesons—we should rather consider $\tilde{\psi} = \exp[g\gamma_5\Phi]\psi$ as a new type of field interacting with itself, even if then the new interaction (L', Equ. (2)) is not simply a transformed version of the original linear PS-PV interaction we started from.

This is lent support by the fact that the original meson field Φ and the associated mass (μ^*) actually have another physical meaning in the closed form expressions (such as G_c') than they have in the linear interaction Lagrangian. In particular, even in the case $\mu^* = 0$ we are encountered with a spread of the support of H_c which causes the indirect interaction between point-nucleons, the range of this interaction being spread over a region of the order $|g|$, $|g| \approx 1/\mu$. It is worth mentioning that the interaction Lagrangian L' (2) actually arises from the term $\bar{\psi}m\psi$ in the Lagrangian of the original spinor field ψ if the contact transform is carried out ([11]). Thus the weight-function $W = \exp[g\gamma_5\Phi]$ in $M \cdot W$ causes the point-like nucleon mass to spread statistically over the space-time domain given by $|x_\nu^2| \lesssim |g^2|$. From this spread we may also understand that the nucleon mass itself determines the strength of the nucleon-nucleon interaction.

It must be concluded that the separation into a Φ- and a ψ-field is rather artificial and that, in particular, $g\Phi(x)$ should be considered as a single unity where, in virtue of the non-invariance under similarity transforms, $|g|$ determines the fundamental co-ordinate unit ([20]).

We remark finally that the quantity $\tilde{\psi} = W \cdot \psi$ is related to (and perhaps identical with) the renormalized nucleon field $\psi_R(x)$ on account of Equ. (34). ψ_R is obviously a non-localizable operator since T_R is so. It is clear, therefore that Dyson's contact transforms cannot be considered as a renormalization mapping of the type suggested by POMERANČUK *et al.* ([19]), which would map the PS-PV theory into a null theory. The formal conclusion $Z_2 = 0$ must be rejected in full.

In addition, the non-renormalizable infinities encountered with in the usual treatment of PS-PV theory, in which the multiple propagators are not rigorously defined, have in principle nothing to do with the renormalization problem. These infinities are removed automatically if the multiple propagators are imbedded into an appropriate vector space. Instead, these infinities are manifestations of the fact that the representation of the actually

non-local theory in terms of localizable quantities (cf. Equ. (6)) has not been treated by the appropriate mathematical methods.

6·6. – We have seen that the indeterminacy in space-time of the light-cone, due to the non-localizability in R_4 of the support of H_c and likewise of the support of T_R, produces an indeterminacy in R_4 of the positions of two (or more) interacting point-nucleons. Thus the interaction causes a statistical spread in R_4 of those particles which in this sense possess a statistical « extention » which, for example, may be described by the weight-function $h_c(z)$ (Equ. (23)). In this sense two point-nucleons also interact through space-like distances, these effects (restricted to regions $|k_\nu^2| \gtrsim |1/g^2|$, $|x_\nu^2| \lesssim |g^2|$) being described by the lack of positive-definiteness due to the oscillatory behaviour of the momentum space operators and by the (statistically interpreted) repulsive-core potential $R(g, \boldsymbol{r})$ (Equ. (33)).

It should be taken into account, however, that the only mathematical objects which refer to the geometry of the system are the series H_c, (19), and T_R, (34). These series define the underlying geometry in terms of their supports if one adopts the point of view that the physical space-time is defined in terms of the interacting particles. Thus, the series H_c, T_R, which describe the topology of the system, are the only objects from which the physical space-time must deduce its properties. From this point of view, the flat Minkowski continuum, in which H_c, T_R have non-local structure, represents merely a mathematical background system.

We may therefore define the points of the « physical » space-time R_4^* by the abstract supports of H_c, T_R. An algebraic definition of the support concept has been given by KÖNIG ([13]); it applies also to H_c, T_R. Since the series H_c, T_R are elements of an abstract space $\mathcal{H}$, the topology of the set of the supports, *i.e.*, that of the « points » of R_4^*, is determined by the topology of the space of the series H with $0 < \varrho < 1$. Those points possess no « inner structure » since they are the elementary objects of R_4^*. Thus in R_4^* the interaction is localizable while in R_4 it is non-localizable. The physical nucleons may then be considered as occupying the points of R_4^*.

7. – Conclusion.

Summing up, it may be stated that neutral non-linear PS-PV theory, renormalized or not, possesses a non-local structure. This non-localizability is due to an intrinsic indeterminacy in space-time of the light-cone and is caused by the infinite set of topologically independent direct interactions which actually makes up PS-PV theory. There is no structure function in the formalism and the indeterminacy of the light-cone can be interpreted in terms of a statis-

tical spread in Minkowski space of the positions of interacting point-nucleons over a region determined by the length $|g$ (originally the coupling constant) which now seems to play the role of the meson Compton wave length.

The methods used here are directly applicable to the equations of LSZ, in the case of interactions of the second kind. However, Wightman's theory of vacuum expectation values requires a generalization in those cases since the momentum space operators are non-tempered distributions. In particular positive-definiteness conditions are restricted to regions beyond a critical length.

Direct interactions of Fermi type, which likewise are of the second kind, deserve further investigations. It seems likely that those interactions can also be treated by the above methods, *i.e.*, by imbedding the multiple propagators, resulting from the ordinary coupling constant expansion of the S-matrix, into the product space. The decisive criterion for the possible non-localizability of those theories is again the order of growth $\varrho > 0$ of the associated momentum space propagators. That all derivative coupling theories belong to the class characterized by $0 < \varrho < 1$ is almost evident. In the model of scalar beta coupling discussed by ARNOWITT and DESER, the order of growth seems to be larger than zero, what whould imply that beta couplings in fact lead to non-local structures. But so far no definite statements have been made.

* * *

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APPENDIX

Existence theorem of König [5].

Let there be given a vector space $K = \{A, B, \ldots; \alpha\}$ which satisfies the following postulates:

(i) K contains all continuous functions (abbr.: cf) $\alpha(x)$. A linear operator A' is defined for all A in K, coinciding with the derivative if $A = \alpha$ has a continuous derivative.

(ii) A product $A \circ B$ (linear in A, B), valued in K, is defined for some A, B, coinciding with the ordinary product if A and B are cf. The product

$\alpha \circ A$ is defined for all A in K, $\alpha = \text{cf}$. If $A \circ B$ is defined then also $(\alpha \circ A) \circ B$ and $A \circ (\alpha \circ B)$ are so. There exists a unit element 1 in K satisfying $1 \circ A = A$ for all A in K, in particular, $1 \circ (A \circ B) = A \circ B$. $(A \circ B)'$ is defined and there holds the product rule $(A \circ B)' = A' \circ B + A \circ B'$ if the right side exists.

(iii) Let Ω be an indefinitely differentiable (one-to-one) mapping of the real x-axis on itself, Ω having a unique inverse. A mapping Ω^* on K be defined, having a unique inverse, such that $(\Omega^* \alpha)(x) = \alpha(\Omega(x))$ for all $\alpha = \text{cf}$. If $A \circ B$ is defined then $\Omega^*(A \circ B) = (\Omega^* A) \circ (\Omega^* B)$ and $(\Omega^* A)' = \Omega' \circ (\Omega^* A')$ for all A in K.

Schwartz' distributions are contained, as a special case, in K. Dirac's δ (in K) is supposed to satisfy $x \circ \delta = 0$ in K.

Theorem: If $\alpha'' \circ \delta$ is defined in K for all continuous functions (cf) $\alpha(x)$, then δ must be zero, $\delta = 0$, if the postulates (i), (ii), (iii) are to remain satisfied.

Particular consequences of this theorem: $\delta \circ \delta$ and $\delta(\sigma) \circ (1/\sigma)$ (with $\sigma \circ \delta = 0$, $\delta \neq 0$) can be defined only in a space which renounces the simultaneous validity of the above axioms. A space of this type is, for example, König's quotient space Π. Note, as an example, that in Π: $\delta_-(\lambda x) \circ \delta_-(\lambda x) \neq \lambda^{-2} \delta_-(x) \circ \delta_-(x)$.

Formulae.

As to the formulae (A.4), (A.5) c.f. References ([22]) and ([23]). The following relations have been used in the text:

$$\text{A.1)} \quad \begin{cases} (2\pi i \delta_-^{(n)}(x) = (-1)^n n! \lim\limits_{\varepsilon \to 0} (x - i\varepsilon)^{-n-1} = (-1)^n n! \,\text{Pf}\,(1/x^{n+1}) + i\pi \delta^{(n)}(x)\,, \\ 2\pi \mathcal{F} \delta_-^{(n)}(x) = (-ik)^n \theta(k)\,, \end{cases}$$

where $\mathcal{F}T = (1/2\pi) \int_{-\infty}^{\infty} \mathrm{d}x \exp[ikx] T$ and $\theta(k) = 1$ if $k > 0$, $\theta(k) = 0$ if $k < 0$.

Let p.v. denote the Cauchy principal value, then (z complex)

$$\text{(A.2)} \qquad \text{Pf}\,(1/x^n)[\varphi(x)] = \text{p.v.} \int_{-\infty}^{\infty} \mathrm{d}z \varphi(z)/z^n\,.$$

The pseudofunctions (Pf) in the following relations are defined in References ([22,23]). Note that the defining functional $\delta^{(n)}(\sigma)[\varphi(\sigma)]$ $(n \geqslant 1)$ contains pseudofunction concepts too.

$$\text{(A.3)} \quad \begin{cases} D_F = (1/2\pi^2) \lim\limits_{\varepsilon \to 0} (\sigma + i\varepsilon)^{-1} = (1/2\pi^2)[\text{Pf}\,(1/\sigma) - i\pi \delta(\sigma)]\,, \\ D_F^{(m)}(\sigma) = \mathrm{d}^m D_F / \mathrm{d}\sigma^m = (-1)^m m!\,(1/2\pi^2)\,\text{Pf}\,(1/\sigma^{m+1}) - (1/2\pi)\delta^m(\sigma)\,. \end{cases}$$

We put $\sigma = r^2 - t^2 = -x_\nu^2$, $x_0 = t$, $k_\nu x_\nu = k_0 x_0 - \boldsymbol{kx}$, $|\boldsymbol{x}| = r$.

Formally,

$$(2\pi)^4 \mathcal{F}T = \int_{-\infty}^{\infty} \mathrm{d}^4x \exp\left[-ik_\nu x_\nu\right] T = T\left(\exp\left[-ik_\nu x_\nu\right]\right).$$

Then

$$\text{(A.4)} \qquad \begin{cases} \mathcal{F}\delta(x_\nu^2) = -(1/4\pi^3)\,\mathrm{Pf}\,(1/k_\nu^2)\,, \\ \mathcal{F}\delta^{(n)}(x_\nu^2) = -\dfrac{(k_\nu^2)^{n-1}[\log|\gamma k_\nu^2/4| - \psi(n)]}{2^{2n+2}\pi^3\,(n-1)!}\,, \end{cases} \qquad n \geqslant 1\,,$$

where $\log\lambda = C = 0.577\ldots$ is Euler's constant and $\psi(n) = \Gamma'(n)/\Gamma(n)$.

Further we have

$$\text{(A.5)} \qquad \begin{cases} \mathcal{F}\,\mathrm{Pf}\,(1/x_\nu^2) = -(1/4\pi)\,\delta(k_\nu^2)\,, \\ \mathcal{F}\,\mathrm{Pf}\,(1/x_\nu^2)^{n+1} = \dfrac{(-1)^{n-1}(k_\nu^2)^{n-1}\varepsilon(k_\nu^2)}{2^{2n+}\,\pi\,n!\,(n-1)!}\,, \end{cases} \qquad n \geqslant 1$$

where $\varepsilon(k_\nu^2) = \mathrm{sign}\,(k_\nu^2)$.

From (A.4) we obtain (with $k_0 = 0$ and Schwartz' formula (VII, 7; 14) [13], c.f., Equ. (14)), $n \geqslant 0$,

$$\text{(A.6)} \qquad \int_{-\infty}^{\infty} \mathrm{d}t\,\delta^n(x_\nu^2) = \frac{\Gamma(n+1/2)}{2\sqrt{\pi}}\,\mathrm{Pf}\frac{1}{r^{2n+1}} - \frac{2\pi[\log\gamma + \psi(n+1/2)]}{2^{2n}(n-1)!}\,\Delta^{n-1}\delta^{(3)}(\boldsymbol{r})\,.$$

The one-dimensional analog to Equ. (15) is (c.f. (II))

$$\text{(A.7)} \qquad [\delta_-(x)\circ\delta_-(x)\circ\cdots\circ\delta_-(x)]_{(n)} = (1/2\pi)^{n-1}\delta_-^{(n-1)}(x)/(n-1)!\,.$$

We have, for ε infinitesimal, up to terms of order $o(\varepsilon)$:

$$\text{(A.8)} \qquad \begin{cases} D_F(\sigma)D_F(\sigma-\varepsilon) \\ = (1/2\pi^2)^2\left\{\mathrm{Pf}\dfrac{1}{\sigma}\,\mathrm{Pf}\dfrac{1}{\sigma-\varepsilon} + \dfrac{i\pi}{\varepsilon}\left[\delta(\sigma) - \delta(\sigma-\varepsilon)\right] - \pi^2\delta(\sigma)\,\delta(\sigma-\varepsilon)\right\} = \\ = (1/2\pi^2)^2\left\{-\dfrac{1}{\varepsilon}\,\mathrm{Pf}\left[\dfrac{1}{\sigma} - \dfrac{1}{\sigma-\varepsilon}\right] + i\pi\delta'(\sigma) + o(\varepsilon) - \pi^2\delta(\sigma)\,\delta(\sigma-\varepsilon)\right\} = \\ = (1/2\pi^2)^2\left\{\mathrm{d}\,\mathrm{Pf}\,(1/\sigma)/\mathrm{d}\sigma + i\pi\delta'(\sigma) + o(\varepsilon) - \pi^2\delta(\sigma)\,\delta(\sigma-\varepsilon)\right\} = \\ = (-1/2\pi^2)\,D_F'(\sigma) - \dfrac{1}{4\pi^2}\,\delta(\sigma)\,\delta(\sigma-\varepsilon) + o(\varepsilon) = \\ = (-1/\pi^2)\,D_F'(\sigma-\varepsilon) - (1/4\pi^2)\,\delta(\sigma)\,\delta(\sigma-\varepsilon) + o(\varepsilon)\,. \end{cases}$$

RIASSUNTO (*)

Si esaminano le teorie quantizzate di campo di seconda specie, le cosiddette teorie non rinormalizzabili. Si è dimostrato recentemente che alcune di queste teorie possono in realtà essere rinormalizzate. Dimostriamo che le teorie di campo di seconda specie, appartenenti agli accoppiamenti per derivazione, hanno struttura non locale. Si discute nella sua forma non lineare la teoria PS-PV neutra del rinculo completo. La non localizzabilità delle teorie sia non rinormalizzate che rinormalizzate è dovuta al fatto che il gruppo infinito delle interazioni dirette topologicamente indipendenti che caratterizza la teoria PS-PV è equivalente a un nuovo tipo d'interazione diretta. La non localizzabilità si manifesta in un'indeterminazione del cono di luce nello spazio-tempo. Tale indeterminazione si può interpretare in termini di una diffusione statistica nello spazio-tempo delle posizioni di due nucleoni puntiformi interagenti in un volume di spazio-tempo la cui estensione è determinata dalla costante d'accoppiamento $|g|$ della versione lineare della teoria PS-PV. La lunghezza $|g|$ perde il suo significato di accoppiamento nel formalismo non lineare in favore della massa del nucleone, ed acquista ora l'ufficio di una costante fondamentale di struttura. In particolare $|g|$ determina la regione in cui il teorema di equivalenza e il potenziale $1/r^3$ perdono la loro validità e cominciano ad agire potenziali di core repulsivi. In virtù del comportamento oscillatorio nella regione di alta energia degli operatori dello spazio non temperato dei momenti si possono imporre condizioni di positività definita solo per regioni $|k_\nu^2| \lesssim |1/g^2|$ e $|x_\nu^2| \gtrsim |g^2|$. La non localizzabilità è concentrata in questi domini e $|g|$ sembra acquistare l'ufficio della lunghezza d'onda Compton del mesone. Si suggerisce una nuova interpretazione della teoria PS-PV non lineare. Si congettura che anche le interazioni diano vita a strutture non locali. Si introducono propagatori multipli per rendere matematicamente significativi nelle teorie di seconda specie gli sviluppi formali delle costanti d'accoppiamento senza produrre grandezze infinite non rinormalizzabili. Si dannargomenti contro la fondatezza matematica del concetto di creazione (distruzione) e di localizzabilità di più di una particella puntiforme in un singolo punto dello spazio-tempo.

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On the Collective Properties of a Boson System.

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Summary. — A method in close correspondence with the theory of superconductivity of Bardeen, Cooper and Schrieffer is applied to a boson system, extending an earlier approach by Bogolubov. Minimizing the energy with respect to a trial ground state vector of exponential form, new equations and expressions are derived for the excitation spectrum of bosons. The expressions are in a one to one correspondence with those for fermions, from which formally they differ only through signs. The physical content of the expressions is, however, very different. Equations with a predominantly repulsive interaction which in the fermion case have no collective solutions, lead to such solutions for bosons because of a partial Bose-Einstein condensation. It is pointed out that the method is related to a linearization of the quantized matter field equations. Temperature dependent equations are obtained by the same methods as in the theory of superconductivity. The λ-point is defined by the disappearance of the partial Bose-Einstein condensation.

1. – Introduction.

The phenomena of superfluidity and superconductivity have many common features. At the same time, the different behaviour of ^{4}He and ^{3}He at low temperatures indicates strong differences in the collective properties of boson and fermion systems. An approach which can be applied both for bosons and fermions has a chance of throwing some light on the analogy and differences in these different systems.

It has been pointed out in a previous paper ([1]), which will be referred to in the following as (F), that the elementary excitations of the theory of super-

([1]) J. G. Valatin: *Nuovo Cimento*, **7**, 843 (1958), referred to in the text as (F).

conductivity of BARDEEN, COOPER and SCHRIEFFER (2) are simply related to new collective fermion variables, the introduction of which greatly simplifies the formal aspects of the theory (3). These fermion variables are analogous to the phonon variables introduced by BOGOLJUBOV (4) to describe the collective behaviour of a boson system. The analogy will be further exploited in the present paper, in which the collective properties of bosons will be investigated by means of the methods applied in the theory of superconductivity, without reference to BOGOLJUBOV's original approximations.

The equations and expressions obtained will all be in close formal analogy to those given in (F), from which they will differ only through signs corresponding to the different statistics of the particles. To bring out this formal analogy and make the comparison easier, the notation will follow that in (F) as closely as possible. Some of the derivations and expressions similar to those given in (F) will be here omitted. At the same time, the additional relationships presented here will have their simple analogue in the fermion case. This applies especially to the simple transformation properties of the operators with real coefficients, and the considerations in connection with the linearization of the matter field equations.

The physical content of the resulting expressions is, however, very different for bosons and for fermions. The occupation numbers which are between zero and one for fermions, can take large values for bosons. Whereas in the case of superconductivity the relevant region of momentum space is a narrow shell around the Fermi surface, in the summations for a boson system the important region is a small sphere around the origin with zero momentum. The long range correlations of electrons in superconductivity result from attractive forces, whereas the main contribution to the superfluidity of helium seems to result from the repulsive part of the interaction. Repulsive interactions which do not lead to collective solutions in the fermion case, will do so for a boson system due to a partial Bose-Einstein condensation peculiar to the statistics, which introduces an inhomogeneous term in the equations.

The derivation of the effective two-body forces is one of the main problems in the theory of superconductivity. At the same time, the simple potential investigated by BARDEEN, COOPER and SCHRIEFFER gives striking agreement with experiments. In liquid helium, the two-body interactions are fairly well-known and one can attempt to find solutions of the equations derived in the present paper for potentials approximating the realistic case. This needs further work. The repulsive potential with constant matrix elements which is investigated here, serves only as an illustration for the explicit solution of

(2) J. BARDEEN, L. N. COOPER, J. R. SCHRIEFFER: *Phys. Rev.*, **108**, 1175 (1957).
(3) See also N. N. BOGOLJUBOV: *Nuovo Cimento*, **7**, 794 (1958).
(4) N. N. BOGOLJUBOV: *Journ. Phys. USSR*, **11**, 23 (1947).

the equations. More realistic potentials are to be used to arrive at quantitative conclusions.

The approximations involved in obtaining the equations from a variational principle and the role of the neglected part of the Hamiltonian also need further investigation.

2. – The trial ground state vector.

The Hamiltonian of the system is a sum of the kinetic energy and of the interaction energies of the particles. As states with an undetermined number of particles will be considered, the role of the Hamiltonian H will be played by the operator $H - \lambda N$, where N is the particle number operator. The chemical potential λ can be combined with the kinetic energy of the particles. Accordingly, one has

$$H - \lambda N = T + V \,, \tag{1a}$$

$$T = \sum_k (\varepsilon_k - \lambda) a_k a_k^+ \,, \qquad \varepsilon_k = \frac{\hbar^2}{2m} k^2 \,, \tag{1b}$$

$$V = \frac{1}{2\Omega} \sum_{k,k',q} V_{k-k'} a_k a_{q-k} a_{q-k'}^+ a_{k'}^+ \,, \tag{1c}$$

where a_k stands for the creation operator, a_k^+ for the annihilation operator of a boson with momentum k. In (1*c*), $V_{k-k'}$, which is assumed real with $V_k = V_{-k}$, is independent of the volume Ω, and differs in this way from the corresponding $V_{kk'}$ in (F) which contains a factor Ω^{-1}.

Bogolubov's 1947 approximation consists in neglecting in the expression (1*c*) for the interaction energy V all terms which do not contain at least two operator factors a_0 or a_0^+ referring to the zero momentum state, and considering a_0, a_0^+ as *c*-numbers (5). The resulting approximate Hamiltonian is then a quadratic expression in the creation and annihilation operators of states with $k \neq 0$, and can be diagonalized by introducing variables of the form

$$\xi_k = \frac{a_k - g_k a_{-k}^+}{(1 - g_k^2)^{\frac{1}{2}}} \,, \qquad \xi_k^+ = \frac{a_k^+ - g_k a_{-k}}{(1 - g_k^2)^{\frac{1}{2}}} \,. \tag{2a}$$

With

$$g_k = g_{-k} \,, \tag{2b}$$

(5) See also, T. D. Lee, K. Huang and C. N. Yang: *Phys. Rev.*, **106**, 1135 (1957); K. A. Brueckner and K. Sawada: *Phys. Rev.*, **106**, 1117 (1957).

these satisfy the commutation relations

$$[\xi_k^+, \xi_{k'}] = \delta_{kk'}, \qquad [\xi_k, \xi_{k'}] = 0, \qquad [\xi_{k'}^+, \xi_k^+] = 0. \tag{2c}$$

In this approximation, ξ_k, ξ_k^+ are the creation and annihilation operators of sound quanta with an energy spectrum

$$E_k = \sqrt{\varepsilon_k(\varepsilon_k + 2\varrho_0 V_k)}, \tag{2d}$$

where $\varrho_0 = h_0/\Omega$ is the number of particles with zero momentum per unit volume.

The method of the present paper is based on the introduction of the same type of collective variables ξ_k without making the same approximations. Instead, the coefficients g_k will be determined as in (F) by minimizing the expectation value of the Hamiltonian with respect to a ground state wave function of a definite form.

As in (F), an algebraic representation of the state vectors will be used, writing unity for the state vector $|0\rangle$ in which no particles are present. For instance, $a_{k_1}a_{k_2}$ will represent a state with two particles having momenta k_1 and k_2. Consider the state vector

$$\Phi_0 = C \exp[A], \tag{3a}$$

with

$$A = \tfrac{1}{2} g_0 a_0^2 + \sum_k g_k a_k a_{-k}, \tag{3b}$$

where in the summation each pair term $a_k a_{-k}$ occurs only once and $k \neq 0$. The factor C is a normalization constant. This state vector Φ_0 is a product of commuting exponential factors $\exp[g_k a_k a_{-k}]$. Separating by a stroke operators and state vectors, one has

$$a_k^+ | \exp[g_k a_k a_{-k}] = g_k a_{-k} \exp[g_k a_k a_{-k}], \tag{3c}$$

$$a_0^+ | \exp[\tfrac{1}{2} g_0 a_0^2] = g_0 a_0 \exp[\tfrac{1}{2} g_0 a_0^2], \tag{3d}$$

from which it follows immediately that

$$\xi_k^+ | \Phi_0 = 0 \tag{3e}$$

for all values of k. It is convenient to define the operators (2a) also for $k = 0$, with $a_{-0} = a_0$, and the commutation relations (2c) hold then for all k. They define a complete orthonormal set of state vectors of the form

$$\Phi_{k_1 \ldots k_j} = (m_{k_1}! \ldots m_{k_j}!)^{-\frac{1}{2}} \xi_{k_1} \ldots \xi_{k_j} | \Phi_0, \tag{4}$$

where m_{k_i} is the number of identical factors ξ_{k_i} in the product (4) and a definite order $k_1 \leqslant \ldots \leqslant k_j$ is assumed.

(3a, b) will be the trial form of the ground state wave function. It corresponds to an indefinite number of particles and is of the same type as the trial function introduced by BARDEEN, COOPER and SCHRIEFFER in the theory of superconductivity. In the fermion case the exponential factor $\exp [g_\varkappa a_\varkappa a_{-\varkappa}]$ reduces to $(1 + g_\varkappa a_\varkappa a_{-\varkappa})$ because $a_\varkappa^2 = 0$.

With the help of the relations

$$a_k = \frac{\xi_k + g_k \xi_{-k}^+}{(1 - g_k^2)^{\frac{1}{2}}}, \qquad a_k^+ = \frac{\xi_k^+ + g_k \xi_{-k}}{(1 - g_k^2)^{\frac{1}{2}}}, \tag{5}$$

which result from (2a), any operator can be expressed in terms of the collective variables ξ_k, ξ_k^+. In the well ordered form, in which creation operators ξ_k stand to the left of annihilation operators ξ_k^+ in each term, the constant term gives the expectation value of the operator in the state Φ_0.

For the number operator $n_k = a_k a_k^+$, one obtains in this way the expectation value

$$h_k = \frac{g_k^2}{1 - g_k^2}, \tag{6a}$$

and for the pair operators $a_k a_{-k}$ and $a_{-k}^+ a_k^+$, the expectation value

$$\chi_k = \frac{g_k}{1 - g_k^2} = \pm \sqrt{h_k (1 + h_k)}. \tag{6b}$$

One has accordingly

$$(1 + 2h_k)^2 - (2\chi_k)^2 = 1. \tag{6c}$$

Contrary to BOGOLUBOV's original paper ([4]) and to (F), and in accordance with more recent work by BOGOLUBOV ([3]), the coefficients g_k in the definitions (2a) have been assumed to be real. Defining ξ_k with a complex g_k, the products $a_k a_k^+$, $a_k a_{-k}$, $a^+{}_k a_k^+$ contain terms with factors $|g_k|^2$, g_k^{*2} and g_k^2. With a real g_k, the transformation of these quantities simplifies, and is given by the equations

$$1 + n_k + n_{-k} = (1 + 2h_k)(1 + \mathcal{N}_k + \mathcal{N}_{-k}) + 2\chi_k (\xi_k \xi_{-k} + \xi_{-k}^+ \xi_k^+), \tag{7a}$$

$$a_k a_{-k} + a_{-k}^+ a_k^+ = 2\chi_k (1 + \mathcal{N}_k + \mathcal{N}_{-k}) + (1 + 2h_k)(\xi_k \xi_{-k} + \xi_{-k}^+ \xi_k^+), \tag{7b}$$

$$n_k - n_{-k} = \mathcal{N}_k - \mathcal{N}_{-k}, \tag{7c}$$

$$a_k a_{-k} - a_{-k}^+ a_k^+ = \xi_k \xi_{-k} - \xi_{-k}^+ \xi_k^+, \tag{7d}$$

where

$$n_k = a_k a_k^+, \qquad \mathcal{N}_k = \xi_k \xi_k^+. \tag{7e}$$

3. – The equations describing the elementary excitations.

Contributions to the expectation value of the Hamiltonian $(1a, b, c)$ in the state Φ_0, or to expectation values in the product states (4), come only from those terms that can be expressed by means of n_k, $a_k a_{-k}$, $a^+_{-k} a^+_k$. Decomposing the Hamiltonian, as in (F), into two parts $H = H_0 + H_1$, where H_1 has vanishing expectation values in the product states (4), (with a similar decomposition $N = N_0 + N_1$), one obtains an expression of the form

$$H_0 - \lambda N_0 = W_0 + \sum_k \tilde{E}_k \mathcal{N}_k + \sum_{k \neq k'} \mathcal{V}_{kk'} \mathcal{N}_k \mathcal{N}_{k'} + \sum_k \mathcal{V}_{kk} (\mathcal{N}_k^2 - \mathcal{N}_k) \,. \tag{8a}$$

Defining the quantities ν_k, μ_k by

$$\nu_k = \varepsilon_k - \lambda + \frac{1}{\Omega} \sum_{k'} (V_{k-k'} + V_0) h_{k'} \,, \tag{8b}$$

$$= \varepsilon_k - \lambda + V_0 \varrho + \frac{1}{\Omega} \sum_{k'} V_{k-k'} h_{k'} \,, \tag{8c}$$

$$\mu_k = - \frac{1}{\Omega} \sum_{k'} V_{k-k'} \chi_{k'} \,, \tag{8d}$$

where $V_0 = (V_k)_{k=0}$, and

$$\varrho = \frac{1}{\Omega} \sum_{k'} h_{k'} = \frac{1}{\Omega} \bar{N} \,, \tag{8e}$$

is the average density of the system, the expectation value W_0 of $H - \lambda N$ in Φ_0 results as

$$W_0 = \tfrac{1}{2} \sum_k \left\{ (\varepsilon_k - \lambda) h_k + \nu_k h_k - \mu_k \chi_k \right\}. \tag{8f}$$

For the excitation energy $\tilde{E}_k$ of the states $\xi_k | \Phi_0$ one obtains

$$\tilde{E}_k = \nu_k (1 + 2h_k) - \mu_k 2 \chi_k \,. \tag{8g}$$

Through $(8b, d)$, $(6a, b)$, W_0 can be considered as a function of the variables g_k which can be chosen to minimize W_0. The equation $\partial W_0 / \partial g_k = 0$ leads to

$$\mu_k g_k^2 - 2\nu_k g_k + \mu_k = 0 \,. \tag{9a}$$

This has the solution

$$g_k = \frac{1}{\mu_k} (\nu_k - E_k) \,, \tag{9b}$$

with

$$E_k = + (\nu_k^2 - \mu_k^2)^{\frac{1}{2}}, \tag{9c}$$

which gives for (6a, b)

$$h_k = \frac{1}{2}\left(\frac{\nu_k}{E_k} - 1\right), \tag{10a}$$

$$1 + 2h_k = \frac{\nu_k}{E_k}, \tag{10b}$$

$$2\chi_k = \frac{\mu_k}{E_k}, \tag{10c}$$

and for (8f, g)

$$W_0 = \tfrac{1}{2} \sum_k \left\{(\varepsilon_k - \lambda)h_k + \tfrac{1}{2}(E_k - \nu_k)\right\}, \tag{10d}$$

$$\tilde{E}_k = \frac{\nu_k^2}{E_k} - \frac{\mu_k^2}{E_k} = E_k. \tag{10e}$$

In choosing the root (9b, c) of the second order equation (9a), it is assumed that $\nu_k > 0$ for all k. As $h_k \geqslant 0$, this choice then follows from (6a). For potentials which lead to $\nu_k < 0$, it follows from (10a), (10e) that one would obtain negative excitation energies E_k, and Φ_0 would not approximate to the ground state.

To determine the quantities ν_k, μ_k, one obtains from (8c, d), (10a, c) the equations

$$\nu_k = \bar{\varepsilon}_k + \frac{1}{\Omega} V_k h_0 + \frac{1}{2\Omega} \sum_{k'}{}' V_{k-k'} \frac{\nu_{k'}}{(\nu_{k'}^2 - \mu_{k'}^2)^{\frac{1}{2}}}, \tag{11a}$$

$$\bar{\varepsilon}_k = \varepsilon_k - \lambda + V_0 \varrho - \frac{1}{2\Omega} \sum_{k'} V_{k-k'}, \tag{11b}$$

$$\mu_k = -\frac{1}{\Omega} V_k \chi_0 - \frac{1}{2\Omega} \sum_{k'}{}' V_{k-k'} \frac{\mu_{k'}}{(\nu_{k'}^2 - \mu_{k'}^2)^{\frac{1}{2}}}. \tag{11c}$$

The reason for separating the term with $k' = 0$, which is to be omitted from the summation $\sum'$, will be apparent later.

The same equations in terms of the unknown quantities $(1+2h_k)$ and $2\chi_k$, which satisfy the relations (6c), can be written with (10b, c) in the form

$$\bar{\varepsilon}_k + \frac{1}{\Omega} V_k h_0 + \frac{1}{2\Omega} \sum_{k'}{}' V_{k-k'}(1 + 2h_{k'}) = E_k(1 + 2h_k), \tag{12a}$$

$$-\frac{1}{\Omega} V_k \chi_0 - \frac{1}{2\Omega} \sum_{k'}{}' V_{k-k'}(2\chi_{k'}) = E_k(2\chi_k), \tag{12b}$$

and the solution of these equations can be formulated as a multilinear problem.

The close formal analogy of the expressions to those given in (F) is apparent in all the equations. The excitation spectrum $E_k = (\nu_k^2 + \mu_k^2)^{\frac{1}{2}}$ of the fermion system, where the collective excitations have an energy gap μ_k, is replaced for the boson system by $E_k = (\nu_k^2 - \mu_k^2)^{\frac{1}{2}}$. This can also be written in the form

$$E_k = [(\nu_k + \mu_k)(\nu_k - \mu_k)]^{\frac{1}{2}} . \tag{13}$$

Besides this connection of the expressions with those of the theory of superconductivity, the formulae contain as a limiting case those of BOGOLUBOV's 1947 paper. The energy spectrum (2*d*) is obtained from (13) by neglecting all occupation numbers h_k except h_0 in (8*b*, *d*) and writing

$$h_k \approx 0 , \quad \chi_k \approx 0 \quad \text{for} \quad k \neq 0 , \qquad \chi_0 \approx h_0 , \tag{14a}$$

$$\nu_k \approx \varepsilon_k + \frac{1}{\Omega} V_k h_0 , \qquad \mu_k \approx -\frac{1}{\Omega} V_k h_0 . \tag{14b}$$

The choice of λ will be justified through the considerations of the next section. The expressions (14*a*, *b*) could also serve as the zeroth approximation in an iteration procedure for solving the equations (11*a*, *c*).

4. – The role of Bose-Einstein condensation, sound velocity, correlation function.

The occupation numbers h_k with $k \neq 0$ will not be negligible in the realistic case corresponding to liquid helium. General arguments have been put forward, however, to show that h_0 and a corresponding one particle state play a special role, with features related to a partial Bose-Einstein condensation ([6,7]). This special role is exhibited also in the present approach.

Disregarding first the possibility of the special role of a one particle state and of $E_0 = 0$, assume that $E_k > 0$ for all k. For the sake of a simple argument, replace $V_{k-k'}$ by a (real) factorizable potential $V_{kk'} = v_k v_{k'}$. With v_k of a definite sign, this represents a predominantly repulsive interaction. Writing (10*c*) in (8*d*) for all k, one obtains

$$\mu_k = -\frac{1}{\Omega} \sum_{k'} v_k v_{k'} \frac{\mu_{k'}}{2E_{k'}} = -\alpha v_k , \tag{15a}$$

$$\alpha = \frac{1}{\Omega} \sum_{k'} v_{k'} \frac{\mu_{k'}}{2E_{k'}} = -\frac{\alpha}{\Omega} \sum_{k'} \frac{v_{k'}^2}{2E_{k'}} . \tag{15b}$$

([6]) See especially O. PENROSE and L. ONSAGER: *Phys. Rev.*, **104**, 576 (1956).

([7]) This condensation plays an important part also in Feynman's theory, R. P. FEYNMAN: *Phys. Rev.*, **91**, 1291 (1953); R. P. FEYNMAN and M. COHEN: *Phys. Rev.*, **102**, 1189 (1956).

Since the last sum in (15b) is positive, (15a, b) has no solution apart from the trivial one, $\alpha = 0$, $\mu_k = 0$. It can be conjectured that the same conclusion holds for the equation

$$\mu_k = -\frac{1}{\Omega}\sum_{k'} V_{k-k'}\frac{\mu_{k'}}{2E_{k'}} \tag{15c}$$

in the case of more general interactions $V_{k-k'}$ of a predominantly repulsive character.

Equations (15a, b, c) are the same for a fermion system, where similar conclusions hold. For fermions one has actually $E_k > 0$ for all k for interactions which lead to collective solutions, because of the energy gap in the spectrum $E_k = (\nu_k^2 + \mu_k^2)^{\frac{1}{2}}$.

For bosons, however, where $E_k = (\nu_k^2 - \mu_k^2)^{\frac{1}{2}}$, one can have $E_0 = 0$, which is required also by general considerations if one wants to interpret the excitation quanta as phonons. With $E_0 = 0$, equations (10a, c) lead to infinite h_0 and χ_0, and the argument connected with (15a, b) does not apply in general.

For a finite volume Ω, the value of $\chi_0 = [h_0(1+h_0)]^{\frac{1}{2}} \approx h_0$ is limited by the total number of particles. Though $h_0 = \infty$ can be satisfied in the limit $\Omega \to \infty$, which is implied in the periodic boundary conditions which introduce a state $k = 0$, but the limit of $\varrho_0 = h_0/\Omega$ remains undetermined by (10a). The term with $k = 0$ in μ_k has, therefore, to be treated separately as in (11a, c).

To determine ϱ_0 and the chemical potential λ, one has two equations, the supplementary condition

$$\varrho = \varrho_0 + \frac{1}{\Omega}\sum_k{}' h_k\,, \tag{15d}$$

and the equation

$$E_0 = 0\,. \tag{15e}$$

It is convenient to determine λ from $E_0 = 0$, and ϱ_0 from (15d). If in the limit of $\Omega \to \infty$, ϱ_0 remains finite, that is if for continuous k the function $h_k/\bar{N}$ contains a δ-function contribution at $k = 0$, there will be a non-trivial solution for μ_k, due to the inhomogeneity introduced by ϱ_0 into the equations. This argument does not assume, however, that ϱ_0 is large in comparison with $\varrho - \varrho_0$.

The collective solutions result in this way for bosons with a predominantly repulsive interaction through an inhomogeneity in the equations due to the condensation of many particles in a single state. For fermions such a Bose-Einstein condensation is not possible, and the equations with the same interaction do not have any collective solutions. The equations of the present approach reflect in this way the different behaviour of ^{4}He and ^{3}He.

With $\nu_k > 0$, $E_k = [(\nu_k + \mu_k)(\nu_k - \mu_k)]^{\frac{1}{2}}$, $E_0 = 0$ leads for a negative μ_k to the equation

$$\nu_0 + \mu_0 = 0 \,. \tag{16a}$$

With a potential $V(r)$ which depends only on the distance r of the particles, the Fourier transform V_k is a function of k^2 only. It can be obtained either as a power series in k^2, or as a limit of such V_k-s. Assuming a power series for V_k, it follows from (8c, d), (16a) and the isotropy in k-space, that

$$\nu_k + \mu_k = \gamma^2 k^2 + \text{higher powers of } k^2 \,. \tag{16b}$$

Near $k = 0$, one has then for E_k

$$E_k = c\,|k|\,, \tag{16c}$$

with

$$c^2 = \gamma^2(\nu_0 - \mu_0) = -\,2\gamma^2\mu_0 \,. \tag{16d}$$

The square of the sound velocity c is in this way proportional to

$$-\,2\mu_0 = \frac{2}{\Omega}\sum_k V_{-k}\chi_k = 2\int V(r)\chi(r)\,\mathrm{d}^3r \,, \tag{16e}$$

where $\chi(r)$ is the Fourier transform of χ_k.

In the case of the energy spectrum (2d), c^2 is proportional to

$$\frac{2}{\Omega}\,V_0\chi_0 = 2\varrho_0\int V(r)\,\mathrm{d}^3r \,. \tag{16f}$$

The function $\chi(r)$ in (16e) is a correlation function factor. It will be seen explicitly in the case of a constant $V_{kk'}$, that the effect of this factor is to make the sound velocity independent of the actual strength of the hard core for a sufficiently large repulsion.

The correlation function in the ground state defined as the expectation value of the operator

$$\Psi^*(x)\Psi^*(x')\Psi(x')\Psi(x) \tag{17a}$$

in the state (3a, b), results from a straightforward calculation as

$$\varrho^2 + h^2(x - x') + \chi^2(x - x') \,, \tag{17b}$$

where $h(r)$ is the Fourier transform of h_k. In the case of a Bose-Einstein condensation, a fluctuation term corresponding to the $k=0$ state is still to be subtracted.

5. – Solution with a simple potential.

BARDEEN, COOPER and SCHRIEFFER have solved the analogous equations of superconductivity with a constant attractive interaction matrix element inside a finite energy interval. Due to the Bose-Einstein condensation, the boson equations have collective solutions for the analogous repulsive interaction. This interaction seems to be less realistic for the helium problem, but provides a simple example for which the solutions can be investigated explicitly.

Accordingly, $V_{k-k'}$ is replaced by

$$V_{kk'} = \begin{cases} V & |k|,\ |k'| \leqslant \varkappa, \\ \text{for} & \\ 0 & |k|,\ |k'| > \varkappa, \end{cases} \tag{18a}$$

with a positive constant V. Equation (11c) leads then to

$$\mu_k = \begin{cases} -\mu & |k| \leqslant \varkappa, \\ \text{for} & \\ 0 & |k| > \varkappa, \end{cases} \tag{18b}$$

where the positive constant μ is to be determined from the equation

$$\mu\left\{1 + \frac{V}{2\Omega}\sum_k{}' \frac{1}{E_k}\right\} = V\varrho_0 . \tag{18c}$$

Here χ_0/Ω has been replaced by $\varrho_0 = h_0/\Omega$, and the summation is over $|k| \leqslant \varkappa,\ k \neq 0$.

From (8c), (16a) and (13) one obtains

$$v_k = \begin{cases} \varepsilon_k + \mu & |k| \leqslant \varkappa, \\ \text{for} & \\ \varepsilon_k - \sigma & |k| > \varkappa, \end{cases} \tag{18d}$$

$$E_k = \begin{cases} [\varepsilon_k(\varepsilon_k + 2\mu)]^{\frac{1}{2}} & |k| \leqslant \varkappa, \\ \text{for} & \\ \varepsilon_k - \sigma & |k| > \varkappa . \end{cases} \tag{18e}$$

(10a) gives

$$(18f) \qquad h_k = \begin{cases} \frac{1}{2}\left\{\dfrac{\varepsilon_k+\mu}{[\varepsilon_k(\varepsilon_k+2\mu)]^{\frac{1}{2}}}-1\right\} & \text{for } |k|\leqslant\varkappa\,,\ k\neq 0, \\ 0 & |k|>\varkappa. \end{cases}$$

That is, states with $|k|>\varkappa$ are not occupied in the ground state Φ_0. The constant σ in (18d, e) results from (8b) as $\sigma=\frac{3}{2}\varrho V-\mu$, if for the non-local potential (18a) one replaces V_0 by V for $|k|\leqslant\varkappa$ and by $\frac{1}{2}V$ for $|k|>\varkappa$, on the basis of considerations analogous to those in (F).

Eliminating ϱ_0 from (18c), with the help of

$$\varrho=\varrho_0+\frac{1}{\Omega}\sum_k{}' h_k\,,$$

and using (18e, f), one obtains

$$(19a) \qquad \mu+\frac{V}{2\Omega}\sum_k{}'\left\{\left(\frac{\varepsilon_k+2\mu}{\varepsilon_k}\right)^{\frac{1}{2}}-1\right\}=V\varrho\,.$$

Replacing $\Omega^{-1}\sum_k{}'\ldots$ by an integral $2\pi\alpha_1\int_0^{\varepsilon_\varkappa}\ldots\varepsilon_k^{\frac{1}{2}}\mathrm{d}\varepsilon_k$ over the variable $\varepsilon_k=\hbar^2k^2/2m$, where the constant α_1 is given by

$$(19b) \qquad \alpha_1=\left(\frac{\hbar^2}{2m}\right)^{-\frac{3}{2}},$$

the integration can be carried out immediately to give

$$(19c) \qquad \mu+\frac{2\pi\alpha_1}{3}V\left\{(\varepsilon_\varkappa+2\mu)^{\frac{3}{2}}-(2\mu)^{\frac{3}{2}}-\varepsilon_\varkappa^{\frac{3}{2}}\right\}=V\varrho\,.$$

Solving this equation for V, it can be seen that V as a function of μ becomes infinite for the value of μ satisfying

$$(19d) \qquad \frac{2\pi\alpha_1}{3}\left\{(\varepsilon_\varkappa+2\mu)^{\frac{3}{2}}-(2\mu)^{\frac{3}{2}}-\varepsilon_\varkappa^{\frac{3}{2}}\right\}=\varrho\,.$$

For values of μ between zero and the solution of (19d) V is positive, for larger values V is negative. For a sufficiently large positive V, for which $\mu\ll V\varrho$, the first term of (19c) can be neglected and μ can be determined from (19d) which is independent of V. The sound velocity is then independent of the actual strength V of the repulsive potential.

The non-local potential (19*a*) corresponds in co-ordinate space to the product of two factors of the form

$$\text{const} \times \frac{\sin \varkappa r - \varkappa r \cos \varkappa r}{(\varkappa r)^3}.$$

Equating the first zero of this factor with a hard core radius $r_0 = 2.56$ Å for helium atoms gives $\varkappa = 1.75 \cdot 10^8$ cm^{-1}. With $m = 6.65 \cdot 10^{-24}$ g, and $\varrho = 2.2 \cdot 10^{22}$ cm^{-3}, the sound velocity $c = (\mu/m)^{\frac{1}{2}}$ results for sufficiently large values of V as about 98 m/s, as compared with the extrapolated value 237 m/s for liquid helium at 0° K. The resulting value of c is not very sensitive to small changes in $\varkappa$. The main reason why the simple potential (18*a*) cannot be expected to give a better value for the sound velocity is that the latter depends according to (16*d*, *e*) on the short range correlations, and should be sensitive to the unphysical shape of the curve h_k given by (18*f*) in which the chosen value of $\varkappa$ means a considerable jump to zero, in cutting off the higher momenta.

6. – The linearized matter field equations.

BOGOLUBOV has pointed out [4] that the frequency spectrum (2*d*) can be obtained also by a linearization of the quantized matter field equations by writing $\Psi(x) = \Omega^{-\frac{1}{2}} a_0 + \vartheta(x)$ in the equations, and neglecting the non-linear terms in $\vartheta(x)$. It might contribute to the understading of the approach to show that the method of the present paper, and that of BARDEEN, COOPER and SCHRIEFFER in the theory of superconductivity, is also related to a simple linearization of the field equations.

In performing a transformation (5) to variables ξ_k, and ordering the terms, the Hamiltonian (1*a*) can be decomposed into two parts

$$T + V = (T + \hat{V}) + V_4, \tag{20a}$$

where V_4 is that part of the interaction energy V which, in the ordered form, contains genuinely four factors ξ_k, ξ_k^+, and $\hat{V}$ is the part of V obtained through contractions. In the form (1*c*) of V, the terms which contribute to $\hat{V}$ are those with $q = 0$, in which factors $a_k a_{-k}$ or $a^+_{-k} a^+_k$ are replaced by their expectation value χ_k, and the terms with $k' = q - k$ or $k' = k$ in which a factor $a_k a^+_k$ is replaced by h_k. Expressed in the original variables a_k, a^+_k, the «linearized» part $T + \hat{V}$ of the Hamiltonian has the simple form

$$T + \hat{V} = \text{const} + \sum \{ \nu_k (a_k a^+_k + a_{-k} a^+_{-k}) - \mu_k (a_k a_{-k} + a^+_{-k} a^+_k) \}, \tag{20b}$$

where v_k, μ_k are defined by (8*b*, *d*), and the summation is not to be extended separately over k and $-k$.

In terms of ξ_k, ξ_k^+, (20*b*) can be written with (7*a-e*) as

$$(20c) \quad T+\hat{V}=W_0+ \\ +\sum_k\left\{\tilde{E}_k(\xi_k\xi_k^+ + \xi_{-k}\xi_{-k}^+)_f + [v_k 2\chi_k - \mu_k(1+2h_k)](\xi_k\xi_{-k}+\xi_{-k}^+\xi_k^+)\right\},$$

where W_0 and $\tilde{E}_k$ are given by (8*f*, *g*). The requirement that the last term in (20*c*) should vanish

$$(20d) \qquad \frac{v_k}{1+2h_k}=\frac{\mu_k}{2\chi_k},$$

together with the relationship (6*c*), is identical with the equations (12*a*, *b*) obtained by minimizing W_0. Obtained in this way, (20*d*) expresses for two-body interactions the diagrammatic condition given for the electron-phonon interaction in the more recent work of BOGOLUBOV [3]. The vanishing of the last term of (20*c*) implies also that corrections to the approximation Φ_0 of the form (3*a*, *b*) of the ground state vector do not contain contributions with the presence of a single pair of excitations $\xi_k\xi_{-k}$, if the coefficients of Φ_0 are determined by minimizing the energy.

The time dependent quantum field equations corresponding to the Hamiltonian (1*a*) can be written in the form

$$(21a) \quad i\hbar\frac{\partial}{\partial t}\Psi(x)=\left(-\frac{\hbar^2}{2m}\Delta-\lambda\right)\Psi(x)+\int \mathrm{d}^3x' V(x-x')\Psi^*(x')\Psi(x')\Psi(x).$$

The above linearization of the Hamiltonian corresponds to replacing the last non-linear term of the equation by

$$(21b) \quad \int \mathrm{d}^3x' V(x-x')\Psi^*(x')\Psi(x')\Psi(x) \simeq \\ \simeq \int \mathrm{d}^3x' V(x-x')\left\{\langle\Psi^*(x')\Psi(x')\rangle_0\Psi(x) + \right. \\ \left. + \langle\Psi^*(x')\Psi(x)\rangle_0\Psi(x') + \langle\Psi(x')\Psi(x)\rangle_0\Psi^*(x')\right\},$$

where the contractions $\langle\ \rangle_0$ mean expectation values with respect to the « vacuum » state Φ_0. These can be expressed with the density $\varrho=\bar{N}/\Omega$ and the Fourier transforms of h_k, χ_k as

$$(22a) \qquad \langle\Psi^*(x')\Psi(x')\rangle_0=\varrho,$$

$$(22b) \qquad \langle\Psi^*(x')\Psi(x)\rangle_0=h(x-x'),$$

$$(22c) \qquad \langle\Psi\ (x')\Psi(x)\rangle_0=\chi(x-x').$$

Introducing

$$(22d) \qquad \nu(x-x') = \left\{ -\frac{\hbar^2}{2m}\Delta - \lambda + V_0\varrho \right\} \delta(x-x') + V(x-x')h(x-x'),$$

$$(22e) \qquad \mu(x-x') = -V(x-x')\,\chi(x-x')$$

the linearized form of (21a) is then

$$(23a) \qquad i\hbar\frac{\partial}{\partial t}\Psi(x) = \int \mathrm{d}^3x' \left\{ \nu(x-x')\Psi(x') - \mu(x-x')\Psi^*(x') \right\}.$$

The Fourier transform of this equation and its adjoint, which can be obtained also from the Hamiltonian (20b),

$$(23b) \qquad \begin{cases} i\hbar\dot{a}_k^+ = \nu_k a_k^+ - \mu_k a_{-k}\,, \\ -i\hbar\dot{a}_{-k} = -\mu_k a_k^+ + \nu_k a_{-k}\,, \end{cases}$$

correspond to field oscillations with the frequency $\pm E_k/\hbar$, with

$$(23c) \qquad E_k = (\nu_k^2 - \mu_k^2)^{\frac{1}{2}}.$$

Analogous linearized expressions result at non-zero temperature, if the contractions are interpreted as expectation values with respect to the statistical operator dealt with in the next section. Introducing this linearization with the help of different contractions one might try to improve on the approximations of the present method.

7. – Temperature dependence and transition temperature.

At temperature T, the statistical operator of the grand canonical ensemble can be approximated in a similar way to (F) by an operator of the form

$$(24a) \qquad U_0 = C_0^{-1} \sum_{j=0}^{\infty} \sum_{k_1 \leq \ldots \leq k_j} w_{k_1} \ldots w_{k_j} P_{k_1 \ldots k_j}\,,$$

which represents independent elementary excitations. $P_{k_1 \ldots k_j}$ is the projection operator of the state $\Phi_{k_1 \ldots k_j}$ given by (4); the normalization constant C_0 is

$$(24b) \qquad C_0 = \mathrm{tr} \sum_{j=0}^{\infty} \sum_{k_1 \leq \ldots \leq k_j} w_{k_1} \ldots w_{k_j} P_{k_1 \ldots k_j} = \prod_k \frac{1}{1-w_k}\,,$$

so that $\mathrm{tr}\, U_0 = 1$.

The average number of phonons in state k is given by

$$\langle \mathcal{N}_k \rangle = \operatorname{tr} \mathcal{N}_k U_0 = \frac{w_k}{1 - w_k} = f_k \,, \tag{24c}$$

where for symmetry reasons it is assumed that $f_k = f_{-k}$.

For the average values of the operators $n_k = a_k a_k^+$, $1 + n_k + n_{-k}$, $a_k a_{-k}$, $a_{-k}^+ a_k^+$, one obtains with (7a-e) and (24c)

$$\langle n_k \rangle = (1 + h_k) f_k + h_k (1 + f_k) = h_k^{(T)} \,, \tag{25a}$$

$$\langle 1 + n_k + n_{-k} \rangle = (1 + 2h_k)(1 + 2f_k) = 1 + 2h_k^{(T)} \,, \tag{25b}$$

$$\langle a_k a_{-k} \rangle = \langle a_{-k}^+ a_k^+ \rangle = \chi_k (1 + 2f_k) = \chi_k^{(T)} \,. \tag{25c}$$

The average value

$$W_0^{(T)} = \langle H - \lambda N \rangle \tag{26a}$$

of the Hamiltonian is then of the form (8f) with h_k, χ_k replaced by $h_k^{(T)}$, $\chi_k^{(T)}$ and ν_k, μ_k by

$$\nu_k^{(T)} = \varepsilon_k - \lambda + \frac{1}{\Omega} \sum_{k'} (V_{k-k'} + V_0) h_{k'}^{(T)} \,, \tag{26b}$$

$$\mu_k^{(T)} = -\frac{1}{\Omega} \sum_{k'} V_{k-k'} \chi_{k'}^{(T)} \,. \tag{26c}$$

The free energy expression

$$W_0^{(T)} - T S_0 \tag{26d}$$

formed with

$$T S_0 = -\beta^{-1} \sum_k \{ f_k \log f_k - (1 + f_k) \log (1 + f_k) \} \tag{26e}$$

can be minimized independently with respect to g_k and f_k. For given g_k, a minimization with respect to f_k gives the boson distribution

$$f_k = \frac{1}{\exp [\beta \tilde{E}_k^{(T)}] - 1} \,, \tag{27a}$$

$$w_k = \exp [-\beta \tilde{E}_k^{(T)}] \,, \tag{27b}$$

with

$$\tilde{E}_k^{(T)} = \frac{\partial W_0^{(T)}}{\partial f_k} = \nu_k^{(T)} (1 + 2h_k) - \mu_k^{(T)} 2\chi_k \,. \tag{27c}$$

The statistical operator (24*a*) can accordingly be expressed by means of the « linearized » Hamiltonian

$$\mathcal{H}_0 = \overline{W}^{(T)} + \sum_k \tilde{E}_k^{(T)} N_k , \tag{28a}$$

$$\overline{W}^{(T)} = W_0^{(T)} - \sum_k \tilde{E}_k^{(T)} f_k , \tag{28b}$$

in the form

$$U_0 = \frac{\exp\left[-\beta \mathcal{H}_0\right]}{\operatorname{tr} \exp\left[-\beta \mathcal{H}_0\right]} . \tag{28c}$$

A minimization of (26*d*), or of $W_0^{(T)}$, with respect to g_k for a given f_k leads to an equation analogous to (9*a*) and to

$$1 + 2h_k = \frac{\nu_k^{(T)}}{E_k^{(T)}} , \tag{29a}$$

$$2\chi_k = \frac{\mu_k^{(T)}}{E_k^{(T)}} , \tag{29b}$$

with

$$E_k^{(T)} = (\nu_k^{(T)2} - \mu_k^{(T)2})^{\frac{1}{2}} , \tag{29c}$$

and

$$\tilde{E}_k^{(T)} = E_k^{(T)} . \tag{29d}$$

The correlation function, the expectation value of the operator (17*a*) in U_0, is obtained from (17*b*) by replacing the Fourier transforms of h_k, χ_k by those of $h_k^{(T)}$, $\chi_k^{(T)}$.

The equations which result by combining (26*b*, *c*), (29*a*, *b*, *c*) and (25*b*, *c*) are very similar to those at $T = 0$. In order to obtain non-trivial solutions in the case of potentials with a repulsive character, one has to determine λ from the condition $E_0^{(T)} = 0$ and $\varrho_0^{(T)} = h_0^{(T)}/\Omega$ from the equation

$$\varrho = \varrho_{(0)}^{(T)} + \frac{1}{\Omega} \sum_k{}' h_k^{(T)} . \tag{30}$$

This introduces the necessary inhomogeneous terms in the equations if $\varrho_0^{(T)}$ remains finite in the limit of large Ω. With increasing temperature $\varrho_0^{(T)}$ decreases, and the λ-point is the temperature at which, in the limit of continuous k, the δ-function contribution to the h_k distribution at $k = 0$ disappears. Above this temperature, which is to be determined from equation (30), there is no Bose-Einstein condensation to add an inhomogeneous term to the equations, and there are no collective solutions.

With the repulsive interaction (18*a*), one concludes $\mu_k^{(T)} = -\mu^{(T)}$ for $|k| \leqslant \varkappa$ and $\mu_k^{(T)} = 0$ for $|k| > \varkappa$. The positive constant $\mu^{(T)}$ which is decreasing with temperature is to be determined from an equation of the form (18*c*), with an additional factor $(1+2f_k)$ under the summation sign, μ, E_k, ϱ_0 replaced by $\mu^{(T)}$, $E_k^{(T)}$, $\varrho_0^{(T)}$, and from equation (30). The energy spectrum $E_k^{(T)}$ is still of the form (18*e*). Near $T = 0$, this gives a specific heat c_v proportional to T^3. Slightly below the transition temperature, one will have $\mu^{(T)} \approx 0$, $E_k^{(T)} \approx \varepsilon_k$ for $|k| \leqslant \varkappa$, but $E_k^{(T)} \approx \varepsilon_k - \sigma^{(T)}$ for $|k| > \varkappa$. Accordingly there are more excitations into states with $|k| > \varkappa$ than in the case of non-interacting particles, and the value 3.2 °K for a free boson gas ([8]) is an upper bound for the transition temperature.

* * *

We wish to express our thanks for many helpful discussions to members of the Department of Mathematical Physics at the University of Birmingham, and especially to Dr. G. V. Chester, Dr. J. R. Schrieffer and Dr. T. D. Schultz.

([8]) F. London: *Superfluids* (London, 1954), Vol. II, p. 58.

RIASSUNTO (*)

Estendendo un precedente tentativo di Bogoljubov si applica a un sistema di bosoni un metodo in stretta corrispondenza con la teoria della superconduttività di Bardeen, Cooper e Schrieffer. Minimizzando l'energia rispetto ad un vettore di prova dello stato fondamentale di forma esponenziale si derivano nuove equazioni e nuove espressioni per lo spettro d'eccitazione dei bosoni. Le espressioni sono in corrispondenza univoca con quelle riferentisi ai fermioni dalle quali differiscono formalmente solo pei segni. Il contenuto fisico delle espressioni è, tuttavia, assai differente. Equazioni con interazione prevalentemente repulsiva che nel caso dei fermioni non hanno soluzione collettiva conducono pei bosoni a soluzioni di tal genere per effetto di una parziale condensazione di Bose-Einstein. Si fa rilevare che il metodo è affine a una linearizzazione delle equazioni del campo quantizzato della materia. Equazioni dipendenti dalla temperatura si ottengono con gli stessi metodi usati in teoria della superconduttività. Il punto λ è definito dalla scomparsa della parziale condensazione di Bose-Einstein.

(*) *Traduzione a cura della Redazione.*

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An Investigation on Plasmas in External Magnetic Fields.

I. Steady State.

G. Schmidt

Israel Institute of Technology - Haifa

(ricevuto il 9 Giugno 1958)

Summary. — The stationary state of a magnetically confined neutral plasma is investigated. For a rare plasma, without particle collisions a method is developed by which a complete set of plasma equations is established. This is based upon the simultaneous solution of Maxwell's equations and the equations of motion of the particles. As application of this method, generalized forms of the pressure balance equations are derived. An alternative treatment of this relation on the basis of the Boltzmann equation leads to a general expression, valid also for cases when collisions are not negligible. As an example the plasma equations are solved for a problem, for which the usual first order theory breaks down.

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1. – Introduction.

The behaviour of a plasma in a given electromagnetic field is described by the Boltzmann equation:

$$\frac{\partial f^\sigma}{\partial t} + \frac{\partial f^\sigma}{\partial \boldsymbol{r}} \boldsymbol{v} + \frac{q^\sigma}{m^\sigma} [\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}] \frac{\partial f^\sigma}{\partial \boldsymbol{v}} = \left(\frac{\partial f^\sigma}{\partial t}\right)_{\text{coll}}, \tag{1}$$

where $f^\sigma(\boldsymbol{r}, \boldsymbol{v}, t)$ is the well known distribution function in the configuration and velocity space, for particles of type σ. The electric and magnetic field strengths $\boldsymbol{E}$ and $\boldsymbol{B}$ are however usually unknown as they are produced partly by the plasma itself. The Maxwell equations on the other hand describe the

electromagnetic field if the charge density ϱ and current density $\boldsymbol{i}$ are known:

$$(2)\qquad \begin{cases} \operatorname{curl} \boldsymbol{H} = \boldsymbol{i} + \dfrac{\partial \boldsymbol{D}}{\partial t}\,, & \operatorname{div} \boldsymbol{B} = 0\,, \\ \operatorname{curl} \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}\,, & \operatorname{div} \boldsymbol{D} = \varrho\,. \end{cases}$$

The connection between these two sets of equations is given by the recognition that both ϱ and $\boldsymbol{i}$ consist of two parts; an external part which is usually given, and an other one originating from the plasma itself. Thus

$$(3)\qquad \begin{cases} \varrho = \varrho_{\text{ext}} + \sum\limits_{\sigma} q^{\sigma} \int f^{\sigma}\, \mathrm{d}^3 v\,, \\ \boldsymbol{i} = \boldsymbol{i}_{\text{ext}} + \sum\limits_{\sigma} q^{\sigma} \int f^{\sigma} \boldsymbol{v}\, \mathrm{d}^3 v\,. \end{cases}$$

When the collision term in Eq. (1) is expressed in terms of the $f^{\sigma} - s$ and their derivatives ([1,2]) Eqs. (1), (2) and (3) constitute a complete set of equations which in principle makes the solution of all classical plasma problems possible. It is however extremely difficult to do this practically in most cases. Therefore various approximations are usually made—depending on the nature of the physical problem—which permit the application of simpler equations.

If the plasma is very dense, and the collision term in Eq. (1) has a dominating role, the plasma can often be handled as a conducting fluid ([3-5]). In this case macroscopic hydrodynamical equations can be applied instead of Eq. (1) while Eq. (3) is substituted by Ohms law for a moving fluid and ϱ is generally neglected. These equations together with Eq. (2) form the equations of magneto-hydrodynamics ([6]). The treatment of these equations is also fairly complicated, solutions being available in most cases only for ideal plasmas of infinite conductivity.

The opposite limiting case is, that of a rare plasma, for which the collision term can be entirely neglected. This is a considerable simplification as the expression for $(\partial f/\partial t)_{\text{coll}}$ is extremely complicated ([2]). The direct solution of Eqs. (1), (2) and (3) is even in this case all too difficult. Several attempts

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([1]) W. P. Allis: *Motion of Ions and Electrons*; *Handb. d. Phys.*, **21**, 383 (1956).

([2]) M. N. Rosenbluth, W. M. McDonald and D. L. Judd: *Phys. Rev.*, **107**, 1 (1957).

([3]) H. Alfvén: *Cosmical Electrodynamics* (Oxford, 1950).

([4]) L. Spitzer: *Physics of Fully Ionized Gases* (New York, 1956).

([5]) R. F. Post: *Rev. Mod. Phys.*, **28**, 338 (1956).

([6]) S. Lundquist: *Ark. f. Fys.*, **5**, 297 (1952).

were made in the last years to reduce the equations to a hydrodynamical form ([7-9]). Although the physical situation here differs considerably from the « real » magneto-hydrodynamical case, these attempts were successful in some limiting cases, *e.g.* when the $\boldsymbol{v} \times \boldsymbol{B}$ term is dominant in the Boltzmann equation. No general method is known however which reduces the general equations to magneto-hydrodynamical equations for a rare plasma.

Another method invented for the treatment of a rare plasma is to consider the microstructure of the plasma and to solve—instead of the Boltzmann equation—the equations of motion simultaneously with the Maxwell equations ([10-11]). This method which emphasizes the physical processes in the plasma, had up till now the disadvantage that it gave only approximate results because particle paths were calculated only by the « first order theory » ([1]).

This meant the assumption that the paths are nearly circular, the gyration being superimposed by a small drift velocity of the guiding center. This limits the validity of this approach to the cases, in which the variation of the magnetic field is small inside a cyclotron orbit.

In the following the stationary state of a plasma confined by a magnetic field is investigated. It will be shown that exact plasma equations can be found for a rare plasma combining the equations of motion with the Maxwell equations. For the sake of simplicity the method is presented for a simple geometrical arrangement, but the way of application to more complicated ones is obvious.

Special attention is paid to the pressure balance equation. This law derived previously by TONKS ([10]) on ground of the first order theory is obtained here in a more general form.

Owing to the importance of this equation we derive it also in a different way, directly from Eqs. (1), (2) and (3), and obtain a generalized law, which is valid for all stationary states, including dense plasmas. We investigate this general equation in some limiting cases.

We show furthermore the reduction of this general treatment of a rare plasma, to the first order approximation.

Finally as an example the plasma equations are solved for a simple case.

In a following paper the non-stationary case will be investigated. As usual charge neutrality is assumed throughout the calculations.

([7]) G. CHEW, M. GOLDBERGER and F. LOW: *Proc. Roy. Soc. London.* A **236**, 112 (1956).

([8]) S. CHANDRASEKHAR, A. N. KAUFMAN and K. M. WATSON: *Ann. of Phys.*, **2**, 435 (1957).

([9]) IU. A. CERKOVNIKOV: *Žu. Èksp. Theor. Fyz.*, **5**, 58 (1957).

([10]) L. TONKS: *Phys. Rev.*, **97**, 1443 (1955).

([11]) E. N. PARKER: *Phys. Rev.*, **107**, 924 (1957).

2. – Plasma equations.

A simple « two dimensional » arrangement is assumed with straight and parallel magnetic field lines pointing in the z direction. There is no variation of magnetic field strength along the y co-ordinate and the $x = 0$ plane is chosen as a plane of symmetry. Some typical particle paths are shown in Fig. 1.

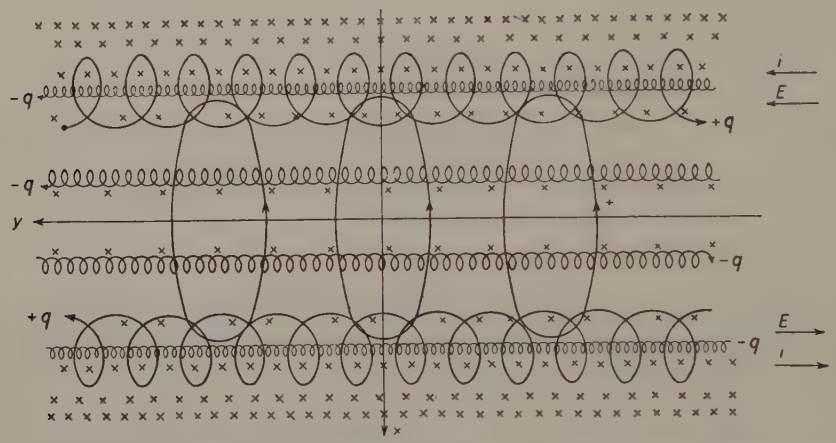

Fig. 1. – Stationary plasma in a magnetic field which arises from outer sources and plasma currents. $+q$ positive ions, $-q$ electrons.

The physical situation may be qualitatively explained as follows.

The outer magnetic field is gradually reduced inside the plasma by the currents arising from the particle motion. These depend on the other side on the distribution of magnetic fields. The actual situation is « self consistent » that is, the magnetic fields arising from the particle motions have just the appropriate distribution, which is needed to maintain the corresponding particle motion. The inhomogeneity of magnetic fields causes average particle velocities opposite for positive and negative particles which correspond to the same direction of current. We see further that not only drifting particles—as in the case of TONKS ([10]) and PARKER ([11])—contribute to the current, but also particles with closed paths. In special cases, a plasma can be confined by a magnetic field without any drifting particles. Note that the actual current density is just opposite to that which one would expect according to the particle drifts of Fig. 1. It has been shown previously ([10]) that the current density is independent of drift velocity, and is a function only of the density gradient. We shall return to this point later.

We form groups, denoting a group, consisting of the same type of particles, with the same velocity v in the x-y plane, and having the same velo-

city components v_x and v_y at the same x co-ordinate, with index i. The velocity of a particle v_i is a constant of motion, the angle α between the velocity vector v_i and the x axis is however a function of x. The magnetic field causes a curvature of the particle path, with a radius of curvature:

$$R_i = \frac{m_i}{q_i} \frac{v_i}{B}. \tag{4}$$

From Fig. 2 it is obvious that

$$R_i \, \mathrm{d}\alpha_i = \frac{\mathrm{d}x}{\cos \alpha_i}, \tag{5}$$

or substituting R_i from (4):

$$\frac{m_i v_i}{q_i \mu H} \cos \alpha_i \mathrm{d}\alpha_i = \mathrm{d}x. \tag{6}$$

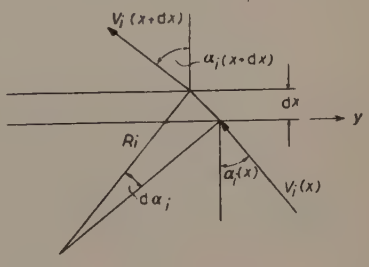

Fig. 2. – Geometrical set-up for computing the particle motion in a perpendicular magnetic field.

The density of particles belonging to the group i is $n_i(x)$ and the net current density in the y direction is:

$$i_y(x) = -\sum_i q_i n_i(x) v_i \sin \alpha_i(x), \tag{7}$$

with $i_x = 0$ assuming $\mathrm{curl}_x \boldsymbol{H} = 0$. For $\mathrm{curl}_y \boldsymbol{H}$ we obtain:

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \sum_i q_i n_i v_i \sin \alpha_i. \tag{8}$$

For each group of particles, there holds an equation of continuity

$$n_i v_i \cos \alpha_i = \lambda_i, \tag{9}$$

where λ_i is the particle flux for the particles of type i.

As every particle moves once in the positive once in the negative x direction, and each path between two turning points belongs to a different group, the number of groups is always even, *e.g.* $2N$. Thus the number of the unknown $n_i(x)$ functions as well as the number of the $\alpha_i(x) - s$ is also $2N$. Therefore the number of functions to be determined, including $H(x)$ equals $4N+1$. As Eqs. (6) and (9) represent both $2N$ equations, these with the additional Eq. (8) suffice for the determination of all unknown functions. Naturally, the same number of boundary values must also be given. The $v_i - s$ are prescribed numbers. The way of generalization of these equations to other two

dimensional geometries (cylindrical etc.) is obvious and will not be discussed here.

A generalized form of the usual pressure balance equation can be simply obtained from the plasma equations without solving them directly. Combining Eq. (8) with (9) and (6) we obtain:

$$\mathrm{d}H = \sum_i \frac{m_i v_i \lambda_i}{\mu H} \sin \alpha_i \mathrm{d}\alpha_i \,. \tag{10}$$

Considering the anisotropic character of particle velocities, the pressure can not be regarded as a scalar. It is a tensor and its x-x component has the form:

$$p_{xx} = \sum_i m_i n_i v_i^2 \cos^2 \alpha_i = \sum_i m_i \lambda_i v_i \cos \alpha_i \,. \tag{11}$$

While the variation of p_{xx} is:

$$\mathrm{d}p_{xx} = -\sum_i m_i \lambda_i v_i \sin \alpha_i \, \mathrm{d}\alpha_i \,. \tag{12}$$

Combining (10) with (12) we find:

$$\mu H \, \mathrm{d}H + \mathrm{d}p_{xx} = 0 \,, \tag{13}$$

or

$$\operatorname{grad} \left(\frac{B^2}{2\mu} + p_{xx} \right) = 0 \,. \tag{14}$$

3. – The pressure balance equation.

As the pressure balance equation is of fundamental importance expressing the confinement and « diamagnetism » of the plasma, we derive it in a general form, for an arbitrary three dimensional geometry including the case of direct particle interactions by collisions.

The behaviour of a plasma is generally described by Eq. (1), which reduces for the steady state to

$$\frac{\partial f^\sigma}{\partial x_\alpha} v_\alpha + \frac{q^\sigma}{m^\sigma} (\boldsymbol{v} \times \boldsymbol{B})_\alpha \frac{\partial f^\sigma}{\partial v_\alpha} = \left(\frac{\partial f^\sigma}{\partial t} \right)_{\mathrm{coll}} , \tag{15}$$

where the component form with the summation convention for common indices is used and we take into account that $\boldsymbol{E} = 0$.

When multiplying Eq. (15) by $m^\sigma v_\beta$ and integrating over velocity space, the first term becomes:

$$\int \frac{\partial f^\sigma}{\partial x_\nu} m^\sigma v_\nu v_\beta \mathrm{d}^3 v = \frac{\partial}{\partial x_\nu} \int f^\sigma m^\sigma v_\nu v_\beta \mathrm{d}^3 v = \frac{\partial}{\partial x_\alpha} (n^\sigma m^\sigma \langle v_\nu v_\beta \rangle^\sigma) , \tag{16}$$

where n^σ is the density of particles of type σ in configuration space. The averages in velocity space are denoted in the usual bracket form.

For the second term, we obtain by integration by parts:

$$q^\sigma \int \frac{\partial f^\sigma}{\partial v_\nu} (\boldsymbol{v} \times \boldsymbol{B})_\nu v_\beta \mathrm{d}^3 v = - q^\sigma \int f^\sigma (\boldsymbol{v} \times \boldsymbol{B})_\nu \frac{\partial v_\beta}{\partial v_\nu} \mathrm{d}^3 v = - q^\sigma \int f^\sigma (\boldsymbol{v} \times \boldsymbol{B}) \delta_{\alpha\beta} \mathrm{d}^3 v , \tag{17}$$

since the surface term vanishes and $(\partial / \partial v_\alpha)(\boldsymbol{v} \times \boldsymbol{B})_\alpha = 0$.

Finally the collision term becomes:

$$\left(\frac{\partial}{\partial t} \int m^\sigma f^\sigma v_\beta \mathrm{d}^3 v \right)_{\mathrm{coll}} = \left(\frac{\partial}{\partial t} m^\sigma n^\sigma \langle v_\beta \rangle^\sigma \right)_{\mathrm{coll}} . \tag{18}$$

Thus we have for the first moment of Eq. (15):

$$\frac{\partial}{\partial x_\nu} (n^\sigma m^\sigma \langle v_\nu v_\beta \rangle^\sigma) - q^\sigma \int f^\sigma (\boldsymbol{v} \times \boldsymbol{B})_\beta \mathrm{d}^3 v = \left(\frac{\partial}{\partial t} m^\sigma n^\sigma \langle v_\beta \rangle^\sigma \right)_{\mathrm{coll}} . \tag{19}$$

The summation over all types of particles yields for the first term of Eq. (19)

$$\frac{\partial}{\partial x_\alpha} \sum_\sigma (n^\sigma m^\sigma \langle v_\alpha v_\beta \rangle^\sigma) = \frac{\partial}{\partial x_\alpha} p_{\alpha\beta} , \tag{20}$$

introducting the pressure tensor $p_{\alpha\beta}$.

Using Eq. (3) and observing that inside the plasma $\boldsymbol{i}_{\mathrm{ext}} = 0$, the second term becomes:

$$- \left(\sum_\sigma q^\sigma \int f^\sigma \boldsymbol{v} \mathrm{d}^3 v \times \boldsymbol{B} \right)_\beta = - (\boldsymbol{i} \times \boldsymbol{B})_\beta . \tag{21}$$

With the aid of Eq. (2) we introduce the magnetic stress tensor in the usual manner

$$- (\mathrm{curl}\, \boldsymbol{H} \times \boldsymbol{B})_\beta = - \frac{\partial}{\partial x_\alpha} T_{\alpha\beta} , \tag{22}$$

where

$$T_{\alpha\beta} = \frac{B_\alpha B_\beta}{\mu} - \frac{B^2}{2\mu} \delta_{\alpha\beta} . \tag{23}$$

Finally the last term in Eq. (19) expresses the momentum gain of the particles as a consequence of collisions. This term vanishes after summation due to the conservation of net momentum in each collision process. We obtain therefore with the help of Eqs. (20) and (22):

$$\frac{\partial}{\partial x_\alpha}(p_{\alpha\beta} - T_{\alpha\beta}) = 0\,. \tag{24}$$

This means, that in the stationary state the divergence of the sum of the material and magnetic pressure tensor vanishes.

In the case of a two dimensional arrangement with $\boldsymbol{B}(0, 0, B_z)$, the magnetic stress tensor reduces to a simple diagonal form. In the two dimensions considered it has the form $T_{lm} = (B^2/2\mu)\delta_{lm}$ and the tensor divergence reduces to the gradient of the scalar $B^2/2\mu$. (We denote the two dimensional Cartesian co-ordinates with latin indices).

The pressure balance equation becomes for this case:

$$\frac{\partial}{\partial x_\alpha} p_{\alpha\beta} + \frac{\partial}{\partial x_\beta}\left(\frac{B^2}{2\mu}\right) = 0\,. \tag{25}$$

In the case of the simple geometry of the previous chapter Eq. (25) reduces to Eq. (14) and to the additional equation:

$$\frac{\partial}{\partial y} p_{xy} = 0\,, \tag{26}$$

which is automatically satisfied with $p_{xy} = 0$ for the particle paths of Fig. 1.

Another example is the case of general two dimensional geometry and scalar material pressure in the two dimensions considered. This is the case usually assumed in magneto-hydrodynamics. In this case $p_{lm} = \delta_{lm} p$ and Eq. (24) reduces to

$$\operatorname{grad}\left(\frac{B^2}{2\mu} + p\right) = 0\,. \tag{27}$$

Eq. (27) yields after integration:

$$\frac{B^2(\xi)}{2\mu} + p(\xi) = \frac{B^2(\infty)}{2\mu}\,, \tag{28}$$

where ξ is the space vector in the x-y plane and $B(\infty)$ the external field which confines the plasma. The scalar pressure is in this case:

$$p = \sum_\sigma n^\sigma m^\sigma \langle v_x^2 \rangle^\sigma = \tfrac{1}{2} \sum_\sigma n^\sigma m^\sigma \langle v^2 \rangle^\sigma = W\,, \tag{29}$$

where W is the kinetic energy density of the plasma. For the third velocity component which is here quite unimportant we have chosen $v_z = 0$. It follows from Eqs. (28) and (29) that:

$$\frac{B^2(\xi)}{2\mu} + W(\xi) = \frac{B^2(\infty)}{2\mu}, \tag{30}$$

that is the total energy density, composed of magnetic and kinetic energy is a constant. It follows further that a perturbation which leaves the $\boldsymbol{B}$ lines straight and parallel, does not change the energy of the system. The plasma is therefore in an indifferent equilibrium position with respect to these kind of perturbations. This fact is important from the point of view of the stability of plasmas. It can be simply shown to agree with the results of ROSENBLUTH and LONGMIRE ([12]).

4. – Reduction to an isotropic velocity distribution.

We return now to the case of chapter 2 to show the transition to the usual first order theory.

In case of a nearly isotropical velocity distribution it is useful to introduce, for each type of particles, *e.g.* for electrons the mean velocity:

$$\boldsymbol{w}_e = \frac{\sum_i n_{ie} \boldsymbol{v}_{ie}}{n_e}, \tag{31}$$

where the index e refers to electrons. Of course the same is valid for each type of ions. In the following treatment we omit these indices. The velocity $\boldsymbol{v}_i$ may be written as a superposition of a random term $\boldsymbol{u}_i$ and $\boldsymbol{w}$:

$$\boldsymbol{v}_i = \boldsymbol{u}_i + \boldsymbol{w}. \tag{32}$$

In the nearly isotropical case $u_i \gg w$ therefore we shall neglect w^2 compared to u_i^2. The average of the random term is on account of Eqs. (31) and (32) obviously zero.

For one type of particles it is from (11):

$$p_{xx} = m \sum_i n_i v_i^2 \cos^2 \alpha_i = m \sum_i n_i v_{ix}^2 \tag{33}$$

([12]) M. N. ROSENBLUTH and C. L. LONGMIRE: *Ann. of Phys.*, **1**, 120 (1957).

or expressing v_{ix} from Eq. (32):

$$p_{xx} = m\left(\sum_i n_i u_{ix}^2 + 2w_x \sum_i n_i u_{ix} + w_x^2 \sum_i n_i\right) = m \sum_i n_i u_{ix}^2 , \tag{34}$$

since the second term vanishes and the third can be neglected. For p_{yy} we get the same result and can write:

$$p_{xx} = p_{yy} = p . \tag{35}$$

The pressure is in this case a scalar and Eq. (14) is reduced to Eq. (27). Let us see now which is the contribution of one type of particle to the plasma current maintaining the magnetic field gradient. From Eq. (7) follows:

$$i_y = -q \sum_i n_i v_i \sin \alpha_i . \tag{36}$$

Differentiating Eq. (9) we obtain:

$$\mathrm{d}n_i v_i \cos \alpha_i - n_i v_i \sin \alpha_i \, \mathrm{d}\alpha_i = 0 . \tag{37}$$

By combining Eqs. (36), (37) and (6) it follows:

$$i_y = -q \sum_i v_i \cos \alpha_i \frac{\mathrm{d}n_i}{\mathrm{d}\alpha_i} = -\frac{m}{B} \sum_i v_{ix}^2 \frac{\mathrm{d}n_i}{\mathrm{d}x} . \tag{38}$$

With help of Eq. (9) it can be shown that

$$\mathrm{d}(v_{ix}^2 n_i) = \lambda_i \mathrm{d}\left(\frac{1}{n_i}\right) = -\frac{\lambda_i^2}{n_i^2} \mathrm{d}n_i = -v_{ix}^2 \mathrm{d}n_i , \tag{39}$$

therefore

$$i_y = \frac{m}{B} \frac{\mathrm{d}}{\mathrm{d}x} \sum_i n_i v_{ix}^2 = \frac{m}{B} \frac{\mathrm{d}}{\mathrm{d}x} \sum_i n_i u_{ix}^2 . \tag{40}$$

As $\boldsymbol{u}_i$ is isotropic $\sum_i n_i u_{ix} = \frac{1}{2} \sum_i n_i u_i^2$. If the velocity distribution is Maxwellian in the x-y plane with the same two dimensional kinetic temperature T everywhere:

$$i_y = \frac{1}{B} \frac{\mathrm{d}}{\mathrm{d}x} \sum_i \frac{1}{2} n_i m u_i^2 = \frac{1}{B} kT \frac{\mathrm{d}n}{\mathrm{d}x} . \tag{41}$$

This means that i_y does not depend on $\mathrm{d}B/\mathrm{d}x$, and is therefore independent from the particle drift, but it is a function of the density gradient. If there is only one particle velocity Eq. (41) reduces to Eq. (17) of TOKNS [10].

If we look for the current flowing in the region above a given $x = x_0$ on Fig. 1 this is composed of two parts. All the particle paths lying entirely in this region correspond to a current directed to $+y$ while those cut by the plane x_0 cause obviously an opposite current, which more then counterbalances the former.

Eq. (41) gives the current density produced by one type of particle, and shows, that this does not depend on the q or m of these particles, only on their density.

Assuming charge neutrality and only one type of ions with a charge of eZ, we find $Zn_{\mathrm{ion}} = n_e$ and

$$i_e = Zi_{\mathrm{ion}}\,, \tag{42}$$

where the suscripts denote electrons and ions respectively. By hydrogen $Z = 1$ and the contribution of electrons equals that of ions.

* * *

I am grateful to Prof. F. OLLENDORFF for his continuous interest in this work and Mr. G. KALMAN for many helpful discussions.

APPENDIX

We illustrate the use of the plasma equations (6), (8) and (9) by solving them for the simple case represented by Fig. 3. No drifting particles are present all paths being closed. The first order theory is obviously unsuitable for this problem, as it does not predict any variation in the field strength and pressure along the x coordinate. A glance at the figure convinces of the existence of currents decreasing the inner magnetic field.

Due to symmetry Eqs. (8), (9) and (6) reduce to:

$$\frac{\mathrm{d}H}{\mathrm{d}x} = qnv \sin\alpha\,, \tag{43}$$

$$nv\cos\alpha = \lambda\,, \tag{44}$$

$$\frac{mv}{q\mu H}\cos\alpha\mathrm{d}\alpha = \mathrm{d}x\,, \tag{45}$$

observing that $\lambda_1 = \lambda_2$; $n_1 = n_2$ and introducing the notations $n = n_1 + n_2$ and $\lambda = \lambda_1 + \lambda_2$. All variables of interest can be easily expressed as a function of α. Combining Eqs. (43), (44) and (45) we obtain:

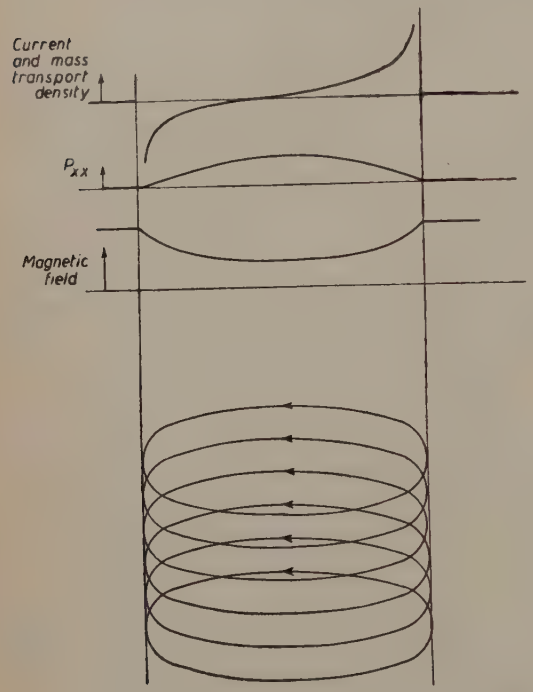

Fig. 3. – Magnetically confined plasma with closed particle paths.

$$(46) \qquad \int_{H_0}^{H} H\,\mathrm{d}H = \frac{\lambda m v}{\mu} \int_{\pi/2}^{\alpha} \sin\alpha\,\mathrm{d}\alpha\,,$$

where H_0 is the external field. The integration yields:

$$(47) \qquad H = H_0 \left[1 - 2\,\frac{q\lambda R_0}{H_0}\cos\alpha\right]^{\frac{1}{2}},$$

introducing $R_0 = mv/qB_0$, the cyclotron radius for vanishing plasma density. In the plane of symmetry:

$$(48) \qquad H(0) = H_0 \left[1 - \frac{2q\lambda R_0}{H_0}\right]^{\frac{1}{2}}.$$

and for the limiting case of $q\lambda R_0 \ll 1$, this becomes reduced to

$$(49) \qquad H(0) \approx H_0 - q\lambda R_0\,.$$

This result can be easily interpreted observing that λq is the mean current density in the y direction and R_0 is the half thickness of the plasma in the limiting case considered.

To express all quantities as functions of x instead of α we substitute Eq. (47) in (45) and integrate:

$$(50) \qquad x = R_0 \int_0^{\alpha} \frac{\cos\alpha\,\mathrm{d}\alpha}{[1 - 2q\lambda R_0 \cos\alpha/H_0]^{\frac{1}{2}}} = -\,R_0 \int_1^{y} \frac{y\,\mathrm{d}y}{\left[(1-y^2)(1-(2q\lambda R_0/H_0)y)\right]^{\frac{1}{2}}},$$

substituting $y = \cos\alpha$. The elliptic integral on the right side can be expressed in terms of Legendre's normal integrals ([13]). With this the problem is essentially solved, as all variables (particle density, field strength, pressure, mass transport density etc.) can now be calculated as functions of x.

([13]) W. Gröbner and N. Hofreiter: *Integraltafel* (Wien, 1949).

RIASSUNTO (*)

Si esamina lo stato stazionario di un plasma neutro contenuto magneticamente. Per un plasma rarefatto senza collisioni di particelle si sviluppa un metodo atto a ricavare un sistema completo di equazioni del plasma. Il metodo è basato sulla soluzione simultanea delle equazioni di Maxwell e delle equazioni del moto delle particelle. Come applicazione del metodo si derivano forme generalizzate delle equazioni dell'equilibrio di pressione. Un trattamento alternativo di questa relazione in base alle equazioni di Boltzmann conduce a un'espressione generale valida anche pei casi in cui le collisioni non sono trascurabili. Come esempio si risolvono le equazioni del plasma di un problema pel quale l'usuale approssimazione del primo ordine cade in difetto.

(*) *Traduzione a cura della Redazione.*

Sulla determinazione del radon nell'aria tellurica mediante lastre nucleari.

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(ricevuto il 17 Giugno 1958)

Riassunto. — Confrontando l'istogramma delle lunghezze delle tracce di particelle α registrate in una lastra nucleare esposta in aria contenente radon in equilibrio con i suoi successori a vita breve, con la distribuzione teorica dovuta alle sole particelle α emesse dagli atomi radioattivi presenti nello strato d'aria antistante la lastra (effetto di volume), viene posta in evidenza la possibilità di separare questo effetto dal contributo delle tracce α emesse dagli atomi del deposito attivo formatosi alla superficie della lastra stessa (effetto di superficie). Ne sono discusse, e controllate in un caso pratico, le possibilità di applicazione al dosamento del radon nell'aria tellurica.

1. – Introduzione.

È stato rilevato, in una nota precedente (1), che quando una lastra nucleare è esposta, a sufficiente distanza dalle pareti, in ambiente contenente Radon ed i successori (a vita breve), la distribuzione delle tracce di particelle α osservate nella lastra (astraendo dall'effetto di zero), è dovuta parte alle tracce α emesse dagli atomi presenti nell'aria (distribuzione da emissione in strato spesso), e parte a quelle provenienti dal deposito attivo formatosi alla superficie dell'emulsione (distribuzione da emissione in strato sottile).

(1) G. Aliverti, A. de Maio, G. Lovera e A. Perilli Fedeli: *Nuovo Cimento*, **4**, 1580 (1956).

Perciò per ottenere una valutazione del contenuto radioattivo nell'aria, da cui dipende direttamente il primo effetto, occorre poter sceverare la prima distribuzione (che verrà indicata come distribuzione di volume) dalla seconda (distribuzione di superficie).

Il calcolo della distribuzione delle lunghezze, nell'emulsione esposta, delle tracce di particelle α emesse da uno strato spesso, con la quale verrà confrontata la distribuzione sperimentale, si esegue con un procedimento ben noto [2], di cui vengono qui richiamati soltanto i risultati essenziali.

Si considerino dapprima particelle α, aventi tutte lo stesso percorso R nell'emulsione nucleare, cui corrisponda un percorso R_a nell'aria in condizioni normali; si ponga

$$k = \frac{R}{R_a}; \tag{1}$$

k è il cosiddetto potere d'arresto dell'aria rispetto all'emulsione (che, per semplicità, si suppone indipendente dall'energia delle particelle α, entro l'intervallo limitato di energia con cui si ha a che fare nella questione presente). Se N è il numero di particelle α emesse, durante il tempo di esposizione della lastra, in 1 cm³ dello strato d'aria, antistante l'emulsione ed a contatto immediato con la sua superficie libera, ed avente spessore non inferiore a R_a, si dimostra facilmente che il numero di tracce, provenienti dall'aria, che attraversano 1 cm² di superficie dell'emulsione, è:

$$n_0 = \frac{NR_a}{4} = \frac{NR}{4k}, \tag{2}$$

e che il numero delle tracce aventi nell'emulsione un percorso residuo non inferiore a r ($r \leqslant R$), è:

$$n(r) = \frac{N}{4k}(R - r). \tag{3}$$

Se invece nell'aria sono presenti atomi radioattivi di specie diversa, ma in equilibrio tra loro, sicchè N sia lo stesso per le varie specie di atomi, ed emettenti particelle α con percorsi R_i nell'emulsione e $R_{a,i}$ nell'aria ($i = 1, 2, \ldots$) ovviamente le (2), (3) diventano:

$$n_0 = \frac{N}{4k}\sum_i R_i, \tag{4}$$

$$n(r) = \frac{N}{4k}\sum_i (R_i - r), \qquad \text{per } R_i - r \geqslant 0, \tag{5}$$

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[2] R. D. Evans: *Phys. Rev.*, **45**, 29 (1934); I. Curie: *Journ. Phys. et Rad.*, **7**, 314 (1946).

dove la sommatoria della (4) si estende a tutte le specie di atomi, con radioattività α, presenti, ossia a tutti i valori di i, e la (5) soltanto ai valori di i per cui è: $R_i - r \geqslant 0$.

Perciò la percentuale di tracce con percorso compreso tra r e r' $(r' > r)$ presenti nella distribuzione di volume dell'emulsione è data da

$$p(r, r') = 100 \frac{\sum_i (R_i - r) - \sum_i (R_i - r')}{\sum_i R_i}, \tag{6}$$

con la restrizione fatta dianzi, che le sommatorie a numeratore si estendono solo alle differenze che risultano positive.

Le formule (4)–(6) valgono, come si è detto, nel caso di atomi appartenenti ad una stessa famiglia radioattiva, ed in equilibrio. Nel caso che gli atomi in questione non appartengano ad una sola famiglia radioattiva, fissa restando per le singole famiglie la condizione dell'equilibrio, le (4) e (5) si compongono, ovviamente, di più termini consimili, uno per ciascuna famiglia.

Le tracce α provenienti dal deposito attivo formatosi alla superficie dell'emulsione danno invece luogo ad una distribuzione a massimi centrati sui valori dei percorsi R_i caratteristici delle varie specie di atomi presenti nel deposito, massimi più o meno allargati in conseguenza, essenzialmente, degli errori di misura, e dello straggling, anche in relazione con eventuali inomogeneità dell'emulsione, dovute sia a fluttuazioni locali, sia ad un effetto di profondità.

2. – Risultati sperimentali e discussione.

Queste considerazioni sono state applicate allo studio della distribuzione delle tracce di particelle α in due lastre, esposte l'una in aria arricchita di Rn e l'altra in aria tellurica.

2·1. – La prima lastra (Ilford C2 ordinaria, 25 mm × 75 mm × 0.1 mm; e precisamente la lastra C facente parte delle lastre del ciclo di ricerche cui si riferisce la nota (1)), è stata anzitutto mantenuta per 19 ore in ambiente saturo di vapore acqueo, in modo che, presumibilmente, raggiungesse, del tutto o quasi, l'equilibrio con le condizioni di umidità alle quali sarebbe stata esposta successivamente. Di poi, la lastra è stata introdotta, e lasciata per 4 ore, in un ambiente chiuso, del volume di 11 litri, contenente circa un litro di acqua proveniente da una sorgente radioattiva dell'Isola d'Ischia, e dalla quale, mediante una energica agitazione, grande parte dell'emanazione era stata liberata nell'aria dell'ambiente stesso: quest'ultima operazione era stata eseguita

4 ore prima dell'introduzione della lastra, onde consentire il raggiungimento dell'equilibrio tra il Rn ed i suoi successori a vita breve. Lo spessore dello strato d'aria antistante la lastra (una diecina di cm) era superiore ai percorsi normali delle particelle α emesse dagli atomi radioattivi presenti.

La densità di tracce aventi almeno un estremo alla superficie dell'emulsione (misurata, come in precedenza [1]), per campi discreti situati ai vertici di un reticolato regolare) è risultata di 761.1 tracce per mm². Da questo dato va detratto l'effetto di fondo, di 11,1 tracce per mm², osservato nella lastra testimonio non esposta ad aria arricchita di Rn: si ha quindi una densità effettiva di 750 tracce per mm².

È stata dipoi misurata la lunghezza di 506 tracce singole, senza alcuna discriminazione sia per l'inclinazione che per il percorso residuo. I risultati, raggruppati in intervalli di 3 μm, sono riportati nella seconda colonna della Tab. I, e nella Fig. 1, spezzata a tratto continuo. D'altro canto, mediante

TABELLA I.

$r \div r'$ (μm)	N. tracce osservate (I)	$p(r, r')$	Distrib. di volume calc. (II)	Distrib. di superficie (I)-(II)
0 ÷ 2	0	6.32	35	—
2 ÷ 5	4	9.47	53	—
5 ÷ 8	22	9.47	53	—
8 ÷ 11	36	9.47	53	—
11 ÷ 14	21	9.47	53	—
14 ÷ 17	57	9.47	53	4
17 ÷ 20	48	9.47	53	(—5)
20 ÷ 23	55	9.47	53	2
23 ÷ 26	48	8.42	47	1
26 ÷ 29	69	5.79	32	37
29 ÷ 32	35	3.16	18	17
32 ÷ 35	17	3.16	18	(— 1)
35 ÷ 38	18	3.16	18	0
38 ÷ 41	24	3.16	18	6
41 ÷ 44	41	0.54	3	38
44 ÷ 47	9	—	—	9
47 ÷ 50	1	—	—	1
50 ÷ 53	0	—	—	0
53 ÷ 56	1	—	—	1
tot.	506		560	116

la (6), è stata calcolata la $p(r, r')$. Per tenere conto dell'allungamento delle tracce, dovuto al maggiore contenuto di acqua presente nell'emulsione esposta in ambiente saturo di vapore acqueo, si sono assunti dei percorsi medi supe-

riori (di circa il 6.5% ([3])) ai percorsi in condizioni d'ambiente normali, e precisamente, arrotondando: (Rn) $R_1 = 25{,}0$ μm; (RaA) $R_2 = 28{,}5$ μm; (RaC') $R_3 = 41{,}5$ μm; (i risultati non dipendono in modo critico da piccole variazioni di questi valori).

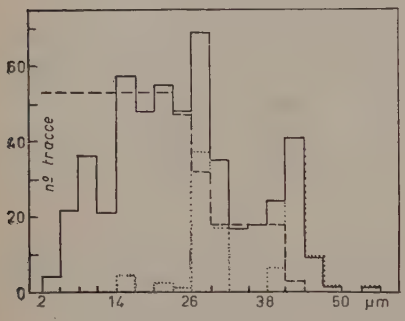

Fig. 1. – Distribuzione delle lunghezze delle tracce di particelle α in una lastra esposta in aria arricchita di Rn: —— distribuzione sperimentale; - - - - - distribuzione di volume, calcolata; ····· distribuzione di superficie.

La distribuzione sperimentale, della lunghezza delle tracce, presenta, rispetto all'andamento calcolato della distribuzione di volume, una carenza di tracce nei primi intervalli: evidentemente tracce assai corte sfuggono all'osservazione, specie se assai inclinate, oppure si confondono con le eterogeneità e le perturbazioni presenti alla superficie libera dell'emulsione. I tre intervalli tra 14 e 23 μm (v. Fig. 1), nei quali da un lato $p(r, r')$ si mantiene costante, e d'altra parte i dati sperimentali oscillano entro limiti piuttosto ristretti, possono fornire un valore di normalizzazione per la distribuzione di volume; essi sono inoltre sufficientemente distanti dal valore della lunghezza delle tracce del RaA, sicchè non c'è da temere un apprezzabile contributo della distribuzione di superficie. La media dei numeri di tracce nei tre intervalli ammonta a 53: facendo corrispondere questo valore al 9.47% della distribuzione di volume calcolata, si ottengono i dati della quarta colonna della Tab. I, arrotondati a valori interi (Fig. 1: spezzata a tratto interrotto). Nella colonna successiva sono riportate, a partire dall'intervallo (14÷17) μm, le differenze tra la distribuzione sperimentale e quella calcolata, relativa all'effetto di volume: questa colonna deve cioè rappresentare la distribuzione delle tracce dovute al solo deposito attivo formatosi alla superficie dell'emulsione. Effettivamente questa distribuzione è costituita sostanzialmente dai due massimi, bene isolati, dovuti al RaA ed al RaC' (Fig. 1: spezzata a punti). In tutta l'estensione della distribuzione di superficie il fondo è praticamente eliminato: ciò convalida la scelta fatta, del valore di normalizzazione per la distribuzione di volume, e, soprattutto, la possibilità di separare le due distribuzioni.

La somma dei dati calcolati per la distribuzione di volume, estesa a tutti gli intervalli, ammonta a 560; inoltre vi sono le 116 tracce del deposito superficiale (somma delle differenze positive della quinta colonna): il che significa che se si fossero misurate tutte le tracce più corte, alle 506 della distribuzione sperimentale se ne sarebbero aggiunte altre 170. Siccome si è determinata

([3]) J. P. LONCHAMP: *Ann. de Phys.*, **10**, 201 (1955).

una densità di tracce di 750 per mm², cioè $7.50 \cdot 10^4$ per cm², aumentando nella stessa proporzione questo dato e detraendo il contributo dell'effetto di superficie, si trova che la densità di tracce, dovuta al solo effetto di emissione di volume, qualora non si perdesse nel conteggio parte delle tracce più corte, dovrebbe ammontare a $8.30 \cdot 10^4$ per cm². In base a questo valore, si può fare una valutazione del contenuto in Rn dell'aria dell'ambiente in cui la lastra è stata esposta.

Detto ν_{Rn} il numero di atomi di Rn presenti, all'inizio dell'esposizione della lastra, in 1 cm³ dell'aria ambiente, e t_e la durata dell'esposizione (nel nostro caso, 4 ore), si ha ovviamente, per il numero delle particelle α emesse:

$$N = \nu_{Rn}(1 - \exp[-\lambda_{Rn} t_e]) . \tag{7}$$

D'altro canto, per quanto si è detto, è:

$$N \frac{\sum_i R_i}{4k} = 8.30 \cdot 10^4 , \tag{8}$$

perciò

$$\nu_{Rn}(1 - \exp[-\lambda_{Rn} t_e]) \frac{\sum_i R_i}{4k} = 8.30 \cdot 10^4 . \tag{9}$$

Sostituendo alle varie quantità presenti nella (9) i loro valori noti, e ponendo

$$\frac{\sum_i R_i}{4k} = \frac{\sum_i R_{a,i}}{4} = \frac{1}{4}(4.05 + 4.66 + 6.91) = 3.90_5 \text{ (cm)} ,$$

si trova: $\nu_{Rn} = 7.15 \cdot 10^5$ atomi per cm³. Il valore che ne deduce, del contenuto in Eman ($4.1 \cdot 10^3$), per l'acqua impiegata, è in accordo soddisfacente con quello ricavato, con altro procedimento di misura, per un secondo campione di acqua della stessa fonte, prelevato quasi contemporaneamente al precedente.

L'esposizione cui è stata assoggettata la lastra in questione riproduce, approssimativamente, le condizioni in cui verrebbe a trovarsi una lastra consimile esposta all'aria tellurica in una buca, praticata nel terreno ed avente raggio tale, che la lastra, collocata in centro, sia circondata da uno strato d'aria di spessore non inferiore al massimo percorso (normale) delle particelle α naturali: questa circostanza è anzitutto una condizione necessaria per i calcoli suesposti, ed inoltre offre anche il vantaggio che, nei riguardi delle tracce α

registrate dalla lastra, entra in gioco solo la radioattività dell'aria ambiente, e non direttamente quella delle pareti della buca.

2'2. – In base a queste considerazioni, per un più immediato controllo del procedimento di elaborazione ora descritto, è stata esaminata con lo stesso criterio un'altra lastra Ilford C2 ordinaria, delle stesse dimensioni, che è stata esposta per 20 ore agli effetti della radioattività dell'aria tellurica, entro una buca praticata nel terreno, e con modalità soddisfacenti alle condizioni geometriche suddette, almeno per il gruppo del Rn (località: Sorrento).

La densità di tracce α, misurata nel modo consueto, è risultata di 39.28 tracce per mm^2; poichè l'effetto di zero su una lastra testimonio ammonta a 1.25 tracce per mm^2, rimane una densità effettiva di 38.0 tracce per mm^2, cioè $3.8 \cdot 10^3$ tracce per cm^2.

Tabella II.

$r \div r'$ (μm)	N. tracce osservate (I)	$p(r, r')$	Distrib. di volume calc. (II)	Distrib. di superficie (I)-(II)
0 ÷ 3	2	9.47	48	—
3 ÷ 6	13	9.47	48	—
6 ÷ 9	14	9.47	48	—
9 ÷ 12	49	9.47	48	1
12 ÷ 15	53	9.47	48	5
15 ÷ 18	44	9.47	48	(— 4)
18 ÷ 21	50	9.47	48	2
21 ÷ 24	46	9.47	48	(— 2)
24 ÷ 27	42	7.38	37	5
27 ÷ 30	69	4.74	24	45
30 ÷ 33	31	3.16	16	15
33 ÷ 36	12	3.16	16	(— 4)
36 ÷ 39	15	3.16	16	(— 1)
39 ÷ 42	17	2.64	13	4
42 ÷ 45	17	—	—	17
45 ÷ 48	13	—	—	13
48 ÷ 51	1	—	—	1
tot.	488		506	108

Nella Tab. II è riportata la distribuzione delle lunghezze su un totale di 488 tracce misurate senza discriminazione di lunghezza (colonna seconda), come pure i risultati dei calcoli eseguiti con la stessa procedura tenuta per la Tab. I, e con gli stessi valori base R_i; i diagrammi corrispondenti sono rappre-

sentati in Fig. 2. Al valore $p(r, r') = 9.47\,\%$ si è fatto corrispondere il numero 48, media delle frequenze osservate nei cinque intervalli compresi tra 9 μm e 24 μm,

Come si vede dall'ultima colonna della Tab. II, la distribuzione di superficie presenta un massimo ben spiccato e localizzato, dovuto al RaA: esso è praticamente isolato, su un fondo di entità trascurabile, tenuto conto delle fluttuazioni statistiche. Quanto al massimo dovuto la RaC′, pure bene isolato, esso è alquanto spostato verso le lunghezze maggiori, e si spinge fino a 48 μm circa. Si è accertato che questo spostamento è dovuto essenzialmente a tracce con decorso superficiale, entro uno strato di spessore non superiore a 10 μm dalla superficie dell'emulsione: evidentemente tale strato presenta un potere d'arresto nettamente inferiore a quello normale dell'emulsione. Un fatto analogo è già stato segnalato ed analizzato da Imbò [4]. Da un esame accurato dei dati si può escludere un contributo apprezzabile da parte del ThC′, le cui particelle α hanno un percorso che, in condizioni normali, è di 47.5 μm, e nel nostro caso, tenuto conto dell'allungamento causato dall'umidità ambiente, salirebbe a circa 50.5 μm. Perciò la elaborazione seguente dei dati è stata eseguita con riferimento soltanto al Rn e successori.

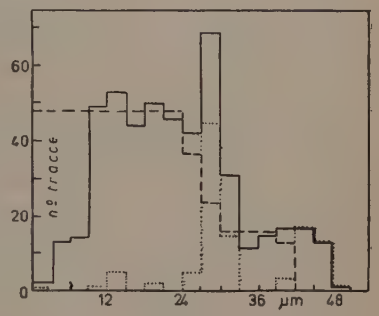

Fig. 2. – Distribuzione delle lunghezze delle tracce di particelle α in una lastra esposta in aria tellurica: —— distribuzione sperimentale; - - - - - distribuzione di volume calcolata; ····· distribuzione di superficie.

Se non ci fosse un difetto di tracce corte osservate, nei primi tre intervalli della Tab. II, l'effetto di volume ammonterebbe da solo a 506 tracce, cui si aggiungerebbero le 108 del deposito superficiale, con un totale di 614 tracce in luogo delle 488 misurate. Essendo la densità totale osservata di $3.8 \cdot 10^3$ tracce per cm², con un calcolo analogo a quello fatto per la lastra precedente si trova che, se non si omettessero tracce corte nel conteggio, la densità di tracce dovuta al solo effetto di volume sarebbe di: $3.9_4 \cdot 10^3$ tracce per cm². Orbene, detto ν_{Rn} il numero di atomi di Rn in un cm³ dell'aria ambiente durante tutta l'esposizione della lastra, con una concentrazione supposta costante, come accade in regime stazionario, dato il rifornimento di Rn dal terreno, e t_e la durata dell'esposizione (20 ore), si ha in questo caso:

$$N = \nu_{Rn} \lambda_{Rn} t_e \,. \tag{10}$$

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[4] G. Imbò: *Ann. Osserv. Vesuviano*, **2**, 64, (1956).

D'altra parte, è

$$N \frac{\sum_i R_i}{4k} = 3.9_4 \cdot 10^3 , \tag{11}$$

quindi

$$\nu_{Rn} \lambda_{Rn} t_e \frac{\sum_i R_i}{4k} = 3.9_4 \cdot 10^3 , \tag{12}$$

e, procedendo in modo analogo al calcolo precedente, e per lo stesso valore di $\sum_i R_i/4k$, si trova: $\nu_{Rn} = 0.67 \cdot 10^4$ atomi per cm^3, valore che è appunto normale per l'aria tellurica, e quindi probativo nei riguardi dell'efficienza del metodo in questione.

3. – Conclusioni.

Il procedimento qui descritto presuppone l'equilibrio radioattivo tra i varí nuclidi di una stessa famiglia. In base ai risultati ottenuti, esso appare soddisfacente per il dosamento del Rn in equilibrio con i suoi successori a vita breve, da soli, cioè quando il Tn ed i suoi successori non siano presenti, almeno in misura rilevante relativamente al gruppo del Rn. Per il gruppo del Tn, la situazione si prospetta differente, in quanto, dati i periodi molto brevi del Tn e del ThA, il processo della loro disintegrazione entra in modo essenziale in competizione con il fenomeno di diffusione nell'aria della buca a partire dal terreno, e quindi c'è da attendersi che la loro distribuzione sia assai disuniforme nell'aria stessa, se essa è in quiete; condizioni di uniformità assai più soddisfacenti dovrebbero invece sussistere per il ThC ed il ThC′, dato il periodo, considerevolmente più lungo, del ThB. Perciò, almeno per il Tn ed il ThA è difficile prevedere adeguatamente la distribuzione teorica delle lunghezze delle tracce dovute all'effetto di volume; ossia la presenza eventuale del Tn e successori è un fattore perturbatore dell'analisi. A questo riguardo, c'è da osservare che la formazione di un deposito attivo alla superficie dell'emulsione, se da un lato complica la distribuzione delle lunghezze delle tracce nella lastra, d'altra parte è utile per riconoscere, in base alla presenza ed alla posizione dei massimi, se insieme al gruppo del Rn intervenga anche quello del Tn. Quest'ultimo può essere segnalato dalla presenza del massimo dovuto al ThC′, il quale, essendo situato all'estremità della distribuzione, dalla parte delle lunghezze maggiori, è il meno disturbato dalla distribuzione delle tracce provenienti dalla emissione di volume. La presenza di questo massimo, e la sua entità, confrontata con quella del massimo dovuto al RaC′ (che si trova

in posizione analoga per il gruppo del Rn), può servire ad indicare se il gruppo del Tn sia presente, ed in misura tale da perturbare l'analisi, o meno.

Nell'ambito di questa limitazione, è lecito affermare che la possibilità, controllata nel presente lavoro, di separare, nella distribuzione delle tracce di particelle α, l'effetto di volume da quello di superficie, e di valutare, di conseguenza, la concentrazione degli atomi radioattivi del gruppo del Rn presenti nell'aria dell'ambiente in cui la lastra è stata esposta, assicura l'idoneità delle lastre nucleari all'impiego come mezzo di prospezione radioattiva del terreno.

SUMMARY

The comparison of the experimental length distribution of α-particle tracks recordee in a nuclear plate exposed in air containing radon in equilibrium with its short-lives decay products, with the theoretical distribution accountable only to the α-particles emitted from the radioactive atoms existing in the air layer facing the plate (volumd effect) enables the discrimination between this effect and the contribution from thd tracks of α-particles emitted from the film of radioactive atoms deposited at the surface of the same plate (surface effect). The possibility of application of this method to the evaluation of the Rn content in telluric air, is discussed and tested with an experimental check.

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Dispersion Relation for Potential Scattering.

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Summary. — A dispersion relation for the scattering of a Klein-Gordon particle by a fixed static interaction is derived. The dispersion relation is valid only for momentum transfers less than a given limit determined by the behavior of the interaction at large distances. The dependence of this limit on the mass spectrum is discussed, and it is found that if the system admits sufficiently exothermic inelastic processes, the derivation fails even in the case of forward scattering.

1. – Introduction.

Recently a rigorous derivation of a dispersion relation for the scattering amplitude of a particle obeying the Schrödinger equation was obtained by Khuri (1). In this derivation the solution to the equation of motion was written down, and the behavior of the scattering amplitude for complex momenta was examined. This method has the advantage that a definite statement can be made about the scattering amplitude at high energies. It has the disadvantage that it is necessary to have a specific solution to the equation,

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(1) N. N. Khuri: *Phys. Rev.*, **107**, 1148 (1957). See also N. N. Khuri and S. B. Treiman: *Phys. Rev.*, **109**, 198 (1958).

and the very complicated treatment of this solution even in the simplest case makes inclusion of inelastic processes virtually impossible. In the problem of establishing dispersion relations in field theory, the « causality » condition (expressed by the commutativity of operators on space-like surfaces) seems to yield almost as much information for this purpose as a solution would, and the application of such a condition would appear to simplify the potential problem.

Because of the « diffusion » character of the Schrödinger equation it does not seem possible to define a causality condition in a useful way, but by noting that except for the relation between energy and momentum the Schrödinger and Klein-Gordon equations are the same, one may approach the former via the latter, in which the causality condition is well defined. Such a trick is only possible in a problem in which no multiple processes are allowed, for it is only then that the charge density and the probability density are the same.

In this paper we shall make use of this trick to rederive Khuri's result as a special case of a dispersion relation for a fixed source (2) by using some of the techniques of field theory. We cannot, of course obtain the high energy limit of the scattering amplitude in this way, and in the derivation we shall for simplicity assume Khuri's result. On the other hand we are able to discuss the modifications which arise as a consequence of allowing the possibility of inelastic processes. We find that for interactions which fall off exponentially, the range of the validity of the result for elastic processes is reduced when exothermic inelastic processes are allowed. This result is analogous to similar limitations noted in recent work in field theory (3).

2. – Derivation.

We take over from field theory the result that the causal scattering amplitudes (retarded and advanced) may be written as follows (4)

$$(1) \qquad \langle k' | M_{R,A} | k \rangle = \pm i \int \mathrm{d}^4x \int \mathrm{d}^4y \exp[-ik'x + iky] \cdot \langle \theta(x_0 - y_0)[j(x), j(y)] \mp \delta(x_0 - y_0)[j(x), \dot{\varphi}(y)] \rangle ,$$

(2) Our problem differs from the simpler cases of fixed source theories considered by KLEIN (A. KLEIN: *Phys. Rev.*, **104**, 1131 (1956)) in that interactions are not limited to take place through a single partial wave.

(3) N. BOGOLJUBOV, B. MEDVEDEV and M. POLIVANOV: *Problems of the Theory of Dispersion Relations*, translated at the Institute of Advanced Study. Also H. BREMMERMAN, R. OEHME and J. TAYLOR: *Phys. Rev.*, **109**, 2178 (1958).

(4) M. L. GOLDBERGER: *Phys. Rev.*, **99**, 979 (1955).

where $j(x)$ is the source of the neutral boson field $\varphi(x)$, defined by

$$(\Box^2 - \mu^2)\varphi(x) = -j(x)\,, \tag{2}$$

k and k' are the initial and final momentum four-vectors of the scattered boson, and all reference to the state of the target, which is fixed and whose energy is arbitrarily set at zero, is omitted. We shall use the conventional co-ordinate system in which

$$\boldsymbol{k} = \boldsymbol{p} - \tfrac{1}{2}\boldsymbol{\Delta}\,; \qquad \boldsymbol{k}' = \boldsymbol{p} + \tfrac{1}{2}\boldsymbol{\Delta} \tag{3}$$

so that for elastic scattering

$$\omega_k = \omega_{k'} = (p^2 + \mu^2 + \tfrac{1}{4}\Delta^2)^{\frac{1}{2}}\,; \qquad \boldsymbol{p}\cdot\boldsymbol{\Delta} = 0\,. \tag{4}$$

The second term in (1) involves an equal-times commutator which is evaluated using the canonical commutation relation

$$[\varphi(x), \dot{\varphi}(y)]_{x_0 = y_0} = i\delta(\boldsymbol{x} - \boldsymbol{y})\,. \tag{5}$$

We obtain for it

$$-i\int \mathrm{d}^4x \int \mathrm{d}^4y \exp\left[-ik'x + iky\right]\delta(x_0 - y_0)i\delta(\boldsymbol{x} - \boldsymbol{y})\left\langle \frac{\delta j}{\delta\varphi}(\boldsymbol{x})\right\rangle = \tag{6}$$

$$= 2\pi\delta(\omega_k - \omega_{k'})\int \mathrm{d}^3x \exp\left[-i\boldsymbol{\Delta}\cdot\boldsymbol{x}\right]\left\langle \frac{\delta j}{\delta\varphi}(\boldsymbol{x})\right\rangle = 2\pi\delta(\omega_k - \omega_{k'})\left\langle \frac{\delta j}{\delta\varphi}(\boldsymbol{\Delta})\right\rangle .$$

For the scattering by a point potential

$$j(x) = -V(\boldsymbol{x})\varphi(x) \tag{7}$$

and this term is $-2\pi\delta(\omega_k - \omega_{k'})V(\boldsymbol{\Delta})$, the Born approximation to the scattering amplitude.

If we define $\mathcal{M}_{B,A}(\omega, \Delta)$ by

$$\langle k' \,|\, M_{B,A} \,|\, k\rangle = 2\pi\delta(\omega_k - \omega_{k'})\mathcal{M}_{B,A}(\omega, \Delta) \tag{8}$$

we obtain

$$\mathcal{M}_B(\omega, \Delta) + V(\Delta) = i\int \mathrm{d}^3x \int \mathrm{d}^3y \exp\left[-\frac{i}{2}\boldsymbol{\Delta}\cdot(\boldsymbol{x}+\boldsymbol{y})\right]\exp\left[-ip\hat{e}\cdot(\boldsymbol{x}-\boldsymbol{y})\right]\cdot \tag{9}$$

$$\cdot \int_0^\infty \mathrm{d}t \exp\left[i\omega t\right]\langle j[(\boldsymbol{x}, t), j(\boldsymbol{y}, 0)]\rangle\,,$$

where $\hat{e}$ is a unit vector in the direction of p.

Similarly

$$(10)\qquad \mathcal{M}_{\Delta}(w, \Delta) + V(\Delta) = -i\int \mathrm{d}^3x \int \mathrm{d}^3y \exp\left[-i\frac{\boldsymbol{\Delta}}{2}\cdot(\boldsymbol{x}+\boldsymbol{y})\right] \exp\left[-ip\hat{e}\cdot(\boldsymbol{x}-\boldsymbol{y})\cdot\right.$$

$$\left.\cdot\int_{-\infty}^{0} \mathrm{d}t \exp\left[iwt\right]\langle[j(\boldsymbol{x}, t), j(\boldsymbol{y}, 0)]\rangle\right.,$$

To examine the symmetry properties of $\mathcal{M}_{R.A}$ it is convenient to expand the commutators by introducing a complete set of eigen-states of the Hamiltonian, $|n\rangle$, with energy E_n.

$$(11)\qquad \mathcal{M}_{R.A} + V(\Delta) = \int \mathrm{d}^3x \int \mathrm{d}^3y \exp\left[-i\frac{\Delta}{2}\cdot(\boldsymbol{x}+\boldsymbol{y})\right] \exp\left[-ip\hat{e}\cdot(\boldsymbol{x}-\boldsymbol{y})\right]\cdot$$

$$\cdot\sum_n \left\{\frac{\langle j(\boldsymbol{x}, 0)|n\rangle\langle n|j(\boldsymbol{y}, 0)\rangle}{E_n - \omega \mp i\varepsilon} + \frac{\langle j(\boldsymbol{y}, 0)|n\rangle\langle n|j(\boldsymbol{x}, 0)\rangle}{E_n + \omega \pm i\varepsilon}\right\}.$$

We assume that there is invariance under three-dimensional rotations. This means that $\mathcal{M}_{R.A}$ can only be a function of Δ^2. By the same token $\mathcal{M}_{R.A}$ cannot depend on $\hat{e}$, so that the two exponential factors $\exp\left[-(i\Delta/2)\cdot(\boldsymbol{x}+\boldsymbol{y})\right]$ and $\exp\left[-ip\hat{e}\cdot(\boldsymbol{x}-\boldsymbol{y})\right]$ can be replaced by the cosines.

It is then clear that

$$(12)\qquad \mathcal{M}_{R}(-\omega, \Delta) = \mathcal{M}_{A}(\omega, \Delta)\,.$$

Another consequence of the assumed invariance is that there are no branch points at $p = 0$, because only p^2 enters into the integrand.

To study the analytic properties of $\mathcal{M}_{R.A}(\omega, \Delta)$ as a function of the complex variable ω, for fixed Δ, we make use of a device which has been applied in the same context in field theory [3], and introduce a parameter λ through the relation

$$(13)\qquad p^2 = \omega^2 + \lambda\,.$$

Ultimately we shall be concerned with the « physical » value for $\lambda = -\mu^2 - \frac{1}{4}\Delta^2$, but at this stage we study $\mathcal{M}_{R.A}$ as functions of ω, Δ and λ.

If we postulate the causality condition

$$(14)\qquad [j(\boldsymbol{x}, t), j(\boldsymbol{y}, 0)] = 0 \quad \text{if} \quad |\boldsymbol{x}-\boldsymbol{y}| > t$$

we see from eq. (9) that $\mathcal{M}_{R.A}(\omega, \Delta, \lambda)$ is an analytic function of ω for $\mathrm{Im}\,\omega > 0$, provided that

$$(15)\qquad \mathrm{Im}\,\omega > |\mathrm{Im}\sqrt{\omega^2 + \lambda}| > 0\,,$$

i.e.

$$\lambda > 0 \,. \tag{16}$$

Similarly $\mathcal{M}_A(\omega, \Delta, \lambda)$ is an analytic function of ω for $\mathrm{Im}\,\omega < 0$ if condition (16) holds.

Consider now the quantity

$$A(\omega, \Delta, \lambda) = -i(\mathcal{M}_A(\omega, \Delta, \lambda) - \mathcal{M}_R(\omega, \Delta\, \lambda)) \tag{17}$$

for real ω. From eq. (11) we find that

$$A(\omega, \Delta, \lambda) = 2\pi \int \mathrm{d}^3x \int \mathrm{d}^3y \exp\left[-i\frac{\Delta}{2}\cdot(\boldsymbol{x}+\boldsymbol{y})\right] \exp\left[-ip\hat{e}\cdot(\boldsymbol{x}-\boldsymbol{y})\cdot\right. \tag{18}$$

$$\left.\cdot \sum \{\delta(E_n - \omega)\langle j(\boldsymbol{x}, 0)\,|\,n\rangle\langle n\,|\,j(\boldsymbol{y}, 0)\rangle - \delta(E_n + \omega)\langle j(\boldsymbol{y}, 0)\,|\,n\rangle\langle n\,|\,j(\boldsymbol{x}, 0)\rangle\}\right].$$

The integrand depends on the nature of the mass spectrum. For potential scattering $E_n > \mu$, and therefore

$$A(\omega, \Delta, \lambda) = 0 \quad \text{for} \quad |\omega| < \mu \tag{19}$$

if there are no bound states, which we shall assume for simplicity. If more complicated processes can take place, this condition is altered. For example in the process

$$\mathrm{K}^- + \mathrm{p} \to \mathrm{K}^- + \mathrm{p}$$

where

$$\mathrm{K}^- + \mathrm{p} \to \Lambda^0 + \pi^0$$

is a possible inelastic process, one would obtain $E_n > \mu^*$, where $\mu^* = m_\Lambda - m_p + m_\pi$. In the process

$$\pi^+ + \mathrm{d} \to \pi^- + \mathrm{d}$$

a possible exothermic inelastic process is

$$\pi^- + \mathrm{d} \to 2\mathrm{n}$$

and here $\mu^* = E_B$, where E_B is the deuteron binding energy.

In general therefore

$$\mathcal{M}_R(\omega, \Delta, \lambda) = \mathcal{M}_A(\omega, \Delta, \lambda) \quad \text{for} \quad -\mu^* < \omega < \mu^* \,. \tag{20}$$

The functions $\mathcal{M}_{R,A}$ are thus different representations of a single function which is analytic in the finite plane with cuts along the real axis (Fig. 1). Because of this property

$$\mathcal{M}(\omega, \Delta, \lambda) + V(\Delta) = \frac{1}{2\pi i} \oint \frac{d\omega'}{\omega' - \omega} [\mathcal{M}(\omega', \Delta, \lambda) + V(\Delta)] . \tag{21}$$

The closed contour is taken to follow the cuts (Fig. 1). In eq. (21) it has been *assumed* that

$$\lim_{|\omega| \to \infty} \{\mathcal{M}(\omega, \Delta, \lambda) + V(\Delta)\} = 0 \tag{22}$$

so that the integral on the right hand side converges. Without more details about the model, we cannot obtain the behavior of the scattering amplitude at infinity. A more general assumption, usually made in field theory is that

$$\lim_{|\omega| \to \infty} \{\mathcal{M}(\omega, \Delta, \lambda) - \sum_{r=0}^{N} C_r(\Delta, \lambda) \omega^r\} = 0; \quad N \text{ finite}$$

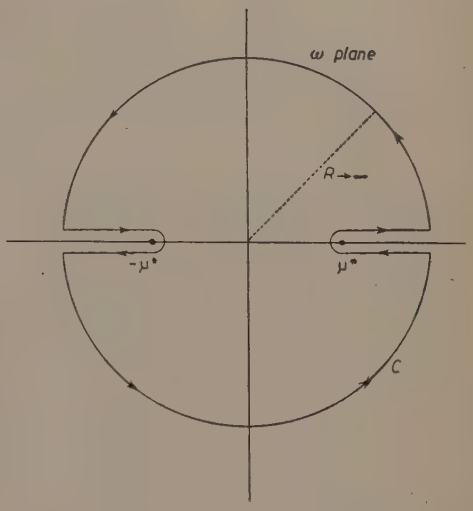

Fig. 1.

but we shall work with (22).

Using

$$\lim_{z \to \pm i\varepsilon} \mathcal{M}(z, \Delta, \lambda) = \mathcal{M}_{R,A}(\omega, \Delta, \lambda) \tag{23}$$

and nothing that the contribution to the integral at infinity vanishes, we obtain with the help of eq. (12)

$$\mathcal{M}(\omega, \Delta, \lambda) + V(\Delta) = \frac{1}{\pi} \int_{\mu^*}^{\infty} \frac{\omega' d\omega'}{\omega'^2 - \omega^2} A(\omega', \Delta, \lambda) , \tag{24}$$

thus establishing the dispersion relations for positive values of the parameter λ. It is now necessary to establish the correctness of eq. (24) for $\lambda = -\mu^2 - \frac{1}{4}\Delta^2$.

As can be seen from (18), the integrand in $A(\omega, \Delta, \lambda)$ is an analytic function of λ for finite complex values of λ. There are no branch points because the whole λ dependence appears in $\cos p\hat{e} \cdot (x - y)$ (cf. discussion following eq. (12)).

If now the interaction falls off exponentially, or more precisely

$$\int \mathrm{d}^3x \exp\left[\alpha|\boldsymbol{x}|\right]|\langle j(\boldsymbol{x},0)|n\rangle| < C \quad \text{for} \quad \alpha < \alpha_0\,, \tag{25}$$

then for a given ω, $A(\omega, \Delta, \lambda)$ will be an analytic function of λ wherever

$$|\mathrm{Im}\sqrt{\omega^2+\lambda}| < \alpha_0\,, \qquad \omega \text{ real}\,. \tag{26}$$

In particular this will be so far

$$|\mathrm{Im}\,\lambda| < \delta\,, \qquad \mathrm{Re}\,\lambda > -\alpha_0^2 - \mu^{*2}\,. \tag{27}$$

Furthermore, for finite λ, the behavior for $A(\omega, \Delta, \lambda)$ as $|\omega| \to \infty$ will be unaltered by the departure from real positive λ's, so that the convergence of the integral on the r.h.s. of eq. (24) is not affected. The r.h.s. of (24) is therefore an analytic function of λ in the region defined by (27), and by analytic continuation the same may be said about $\mathcal{M}(\omega, \Delta, \lambda) + V(\Delta)$. In particular, analytic continuation to $\lambda = -\mu^2 - \frac{1}{4}\Delta^2$ is permitted provided that

$$\alpha_0^2 > \mu^2 - \mu^{*2} + \tfrac{1}{4}\Delta^2\,. \tag{28}$$

In field theory the statement of eq. (25) is not given. The possibility of λ continuation must be derived from a detailed study of the consequences of Lorentz invariance and the causality condition [5].

For the case of potential scattering we may write

$$\langle j(x,0)|n\rangle = (2E_n)^{-\frac{1}{2}} V(|\boldsymbol{x}|)\,\psi_{E_n}(\boldsymbol{x})\,, \tag{29}$$

wherewhere $\psi_{E_n}(\boldsymbol{x})$ is a solution of the wave equation

$$(\nabla^2 + E_n - V(|\boldsymbol{x}|))\psi_{E_n}(\boldsymbol{x}) = 0\,. \tag{30}$$

If the potential is not too singular at the origin, we may choose as a complete set those $\psi_{E_n}(\boldsymbol{x})$ which, as $|x| \to \infty$ have the asymptotic behavior

$$\psi_n(\boldsymbol{x}) \to \exp\left[i\boldsymbol{q}\cdot\boldsymbol{x}\right] + f\frac{\exp\left[iq|\boldsymbol{x}|\right]}{|\boldsymbol{x}|}; \qquad q^2 = E_n\,. \tag{31}$$

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[5] See ref. [3] and G. KÄLLÉN and A. WIGHTMAN: *The Analytic Properties of the Vacuum Expectation Value of a product of Three Scalar Local Fields* (to be published).

Certainly a sufficient condition for this will be that $V(r)$ is not as singular as $1/r^2$ at the origin, and that [6]

$$\int_0^\infty r|V(r)|\,\mathrm{d}r < C' . \tag{32}$$

Since for large $|\boldsymbol{x}|$, $|\psi_{E_n}(\boldsymbol{x})| < 1 + O(1/|\boldsymbol{x}|)$, eq. (25) is equivalent to the requirement that

$$\int_0^\infty \mathrm{d}^3x \exp\left[\alpha|\boldsymbol{x}|\right]|V(\boldsymbol{x})| < C''; \qquad \alpha < \alpha_0 . \tag{33}$$

Since in this case $\mu^* = \mu$, the dispersion relation (24) may be rewritten in terms of $k = \sqrt{\omega^2 - \mu^2}$

$$\mathcal{M}(k, \Delta) + V(\Delta) = \frac{1}{\pi}\int_0^\infty \frac{k'\,\mathrm{d}k'}{k'^2 - k^2}\,\Delta(k', \Delta) ,$$

and it holds provided that

$$\alpha_0^2 > \tfrac{1}{4}\Delta^2 . \tag{35}$$

This result is in agreement with that obtained by KHURI. We again stress that the behavior of the scattering amplitude at infinity is not obtainable by this procedure, and that it had to be assumed. If a more general dependence of the scattering amplitude is obtained (or assumed), the derivation is essentially unaltered except for some minor complications with the continuation [3]. We might add that the identical procedure can be used to find the dispersion relation for the scattering of a Dirac particle by a potential, except that the causality condition is expressed in terms of anticommutators of the current, and attention has to be paid to antiparticle scattering, which will for example modify eq. (12). The last complication also occurs in the case of charged boson scattering, but again a trivial modification in the derivation is all that is necessary [4].

We may just mention here that our technique also allows us to reproduce Khuri's result on the range of validity of the partial wave expansion of the potential scattering amplitude in the non-physical region $0 < \sqrt{\omega^2 - \mu^2} < \frac{1}{2}\Delta$ (see Appendix).

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[6] R. JOST and A. PAIS: *Phys. Rev.*, **82**, 840 (1951).

3. – Conclusions.

The derivation presented in Sect. 2 used known methods to derive results, most of which are not new. The justification for presenting it in such detail is the hope that it brings out the power and elegance of the field theory methods more clearly within the context of a simple problem. The results on the effects of possible inelastic processes merit a few words. First, there is the question whether in the case of an exponentially decreasing interaction, our inability to continue from $\lambda > 0$ to $\lambda = -\mu^2 - \frac{1}{4}\Delta^2$ is a mathematical failure or represents a real property of the scattering amplitude for large enough momentum transfers. Second, we may ask what, if anything, these limitations have to do with the somewhat similar limitations in the analyticity properties of scattering amplitudes in field theory. One might argue that certain field theory interactions, *e.g.* the φ_π^4 term in the pion-nucleon problem or the $\varphi_\pi^2\varphi_K^2$ term in the K-meson-nucleon problem may act like a potential of range $(2\mu)^{-1}$ under certain circumstances and that therefore the similarity between the results of reference [3] and ours is not purely accidental. If this similarity is taken seriously and one assumes the difficulty is not purely mathematical, one would expect the meson-nucleon scattering amplitude to obey the dispersion relation only for $\Delta < 4\mu$ and in the case of K-meson-nucleon scattering ($\alpha_0 = 2\mu_\pi$; $\mu = \mathcal{M}_K$; $\mu^* = \mathcal{M}_\Lambda - \mathcal{M}_p + \mu_\pi$) one would expect to find difficulties even with the forward scattering dispersion relations.

* * *

We wish to thank M. A. RUDERMAN for many stimulating discussions on this and related topics, in particular for suggesting this approach to the Schrödinger problem. One of us (S.G.) wishes to express his appreciation to Professor NIELS BOHR and the staff of the Institute for Theoretical Physics in Copehagen for their kind hospitality.

APPENDIX

In practical applications of dispersion relations it is usual to express the absorptive part of the scattering amplitude $A(\omega, \Delta)$ in terms of experimentally accessible data, *e.g.* phase shifts [7]. The phase shift expansion is not neces-

[7] See, for example, G. CHEW, M. L. GOLDBERGER, F. LOW and Y. NAMBU: *Phys. Rev.*, **106**, 1337 (1957).

sarily valid for « nonphysical » scattering angles,

$$|\cos\theta| = 1\Big| -\frac{1}{2}\frac{\Delta^2}{\omega^2-\mu^2} > 1\,, \tag{A.1}$$

and such a non-physical region always occurs when $\Delta \neq 0$. To examine the validity of the expansion

$$A(\omega,\Delta) = \sum_{l=0}^{\infty}(2l+1)P_l(\nu)A_l(\omega)\,, \tag{A.2}$$

where

$$\nu = 1 - \frac{1}{2}\frac{\Delta^2}{\omega^2-\mu^2}\,,$$

we use the theorem which guarantees the convergence of the expansion (A.2) in an ellipse in the complex ν-plane, the ellipse having foci at ± 1, and extending to the nearest singularity of $A(\omega,\Delta)$ as a function of the complex variable ν (8). To find the size of this ellipse, we examine $A(\omega,\Delta)$. From eq. (18) we see that the Δ dependence of $A(\omega,\Delta)$ is contained in the factor $\exp[-(i/2)\boldsymbol{\Delta}\cdot\cdot(\boldsymbol{x}+\boldsymbol{y}) - i\sqrt{\omega^2-\mu^2-\frac{1}{4}\Delta^2}\hat{e}(\boldsymbol{x}-\boldsymbol{y})]$. Using eq. (25), one may show that $A(\omega,\Delta)$ can be continued to complex values of Δ provided that

$$\tfrac{1}{2}|\operatorname{Im}\Delta| + |\operatorname{Im}\sqrt{\omega^2-\mu^2-\tfrac{1}{4}\Delta^2}| < \alpha_0\,. \tag{A.3}$$

If we specialize to the case of potential scattering $\omega^2-\mu^2 = k^2 \geqslant 0$. With the notation

$$\frac{(\operatorname{Re}\Delta)^2}{2k^2} = x^2; \qquad \frac{(\operatorname{Im}\Delta)^2}{2k^2} = y^2; \qquad \frac{\alpha_0}{k} = \xi\,,$$

condition (A.3) becomes

$$\frac{|y|}{\sqrt{2}} + \left\{\frac{1}{2}\sqrt{\left(1-\frac{x^2-y^2}{2}\right)^2 + x^2y^2} - \frac{1}{2}\left(1-\frac{x^2-y^2}{2}\right)\right\}^{\frac{1}{2}} < \xi\,. \tag{A.4}$$

The equation for an ellipse in the ν-plane, having foci at ± 1 and major axis a is

$$\frac{\nu_{\mathrm{R}}^2}{a^2} + \frac{\nu_{\mathrm{I}}^2}{a^2-1} = 1 \qquad (\nu = \nu_{\mathrm{R}} + i\nu_{\mathrm{I}})\,, \tag{A.5}$$

(8) E. T. Whittaker and G. N. Watson: *Modern Analysis*, p. 322.

which in terms of x and y becomes

$$\frac{x^2}{a+1}+\frac{y^2}{a-1}=1\,. \tag{A.6}$$

To find for what values of a the points inside the above ellipse obey condition (A.4) it is convenient to map the interior of the ellipse onto a strip (Fig. 2). With

$$x=\sqrt{a+1}R\cos\psi;\qquad y=\sqrt{a-1}R\sin\psi\,, \tag{A.7}$$

the interior of the ellipse corresponds to $R\leqslant 1$.

The « boundary » relation (A.4) becomes

$$\sqrt{\frac{a-1}{2}}R|\sin\psi|+\left\{\frac{1}{2}\sqrt{\left(1-\frac{R^2}{2}(a\cos 2\psi+1)\right)^2+\frac{a^2-1}{4}R^4\sin^2 2\psi}-\right. \tag{A.8}$$
$$\left.-\frac{1}{2}\left(1-\frac{R^2}{2}(a\cos 2\psi+1)\right)\right\}^{\frac{1}{2}}<\xi.$$

When we set $R=1$, this becomes

$$\sqrt{\frac{a-1}{2}}(|\sin\psi|+|\cos\psi|)<\xi\,, \tag{A.9}$$

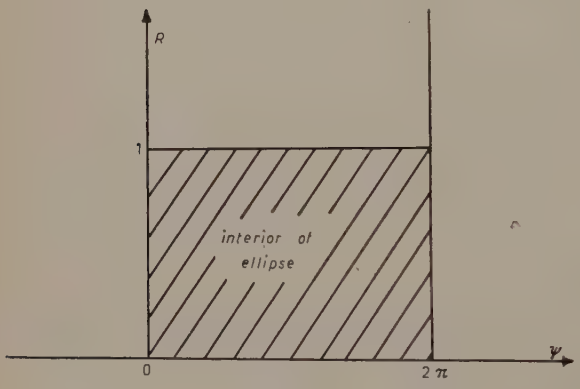

Fig. 2.

so that the « boundary » is always outside the ellipse provided

$$a<1+\xi^2\,. \tag{A.10}$$

Thus analytic continuation by a Legendre polynomial expansion is permitted provided

$$|\cos\theta|<1+\frac{\alpha_0^2}{k^2}\,.$$

In terms of momentum transfers, this condition is

$$\Delta<\sqrt{2}\alpha_0\,. \tag{A.11}$$

This result appears to be an improvement over that of Khuri, who obtains $\Delta<\alpha_0$. The difference is deceptive, because the scattering amplitude also contains the term $V(\Delta)$ (we have only been discussing the right hand side of eq. (34)), and it is easy to see that this term is analytic in Δ for $\Delta<\alpha_0$ only.

It is disappointing that we were unable to obtain the result $\Delta < 2\alpha_0$, for the absorptive part necessary for analytic continuation for all momentum transfers for which the dispersion relation has been established (9).

(9) In the problem of the relativistic field theory of meson-nucleon scattering H. LEHMANN (to be published) has succeeded in showing that there the analytic continuation of the absorptive part can be made for the same momentum transfers for which the dispersion relation was established.

RIASSUNTO (*)

Si deriva una relazione di dispersione per lo scattering di una particella di Klein-Gordon da parte di una interazione statica fissa. La relazione di dispersione è valida solo per trasferimenti di impulso inferiori a un dato limite determinato dal comportamento dell'interazione alle grandi distanze. Si discute la dipendenza di tale limite dallo spettro di massa e si trova che se il sistema ammette un numero sufficiente di processi anelastici isotermici la derivazione viene a mancare anche nel caso di scattering in avanti.

(*) *Traduzione a cura della Redazione.*

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On the Decay of ^{185}W.

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(ricevuto il 23 Giugno 1958)

Summary. — The γ spectra from an electromagnetically enriched sample of ^{184}W irradiated with pile neutrons was investigated. The decay of the various radiations was followed for more than three half-lives of ^{185}W. It was ascertained that the decay of ^{185}W populates the first rotational level of ^{185}Re at 125 keV with a branching ratio of about 0.3%. The K internal conversion of the γ-ray was found to be: 2.9 ± 0.3.

1. – Introduction.

The decay of ^{185}W was investigated by us a few years ago (1): the results then suggested that the β-rays decayed not only to the ground state of ^{185}Re, but even to an excited level at 55.6 keV with a branching ratio of about 10%. While referring the reader to that paper for any previous information on the decay of ^{185}W, we would summarize here the results of the most interesting investigations since carried out by several authors on the matter (2).

Bhattacherjee and Raman (3), by using scintillation spectrometers, found β-rays time coincident with a γ-ray at 56 keV and concluded that the decay was complex with a branching ratio of about 15%. Kreger *et al.* (4) pointed out the presence of two faint γ-rays at 77 and 570 keV which decayed with about the same half-life as the β-ray and of X-rays at 59.5 keV assigned to the decay of ^{181}W contained as impurity in the sample. A faint weakly con-

(1) A. Bisi, S. Terrani and L. Zappa: *Nuovo Cimento*, **1**, 291 (1955).

(2) D. Strominger, J. M. Hollander and G. T. Seaborg: *Rev. Mod. Phys.*, **30**, 585 (1958).

(3) S. K. Battacherjee and S. Raman: *Nuovo Cimento*, **3**, 1131 (1956).

(4) W. E. Kreger, L. D. McIsaac, J. L. Mackin and J. R. Lai: *Phys. Rev.*, **100**, 955 (1955).

verted γ-ray at 132 keV was observed by SEBASTIAN and WEBER (5). No γ-rays were reported by MARTY and VERGNES (6) to be emitted in the decay of ^{185}W; in particular, they attributed the peak at 57 keV to the K electron capture decay of ^{181}W. Furthermore their γ spectra show the presence of a broad peak at 125 keV which was attributed to internal and external bremsstrahlung radiations and a faint peak at 590 keV attributed to unknown impurities.

An investigation using a double focusing β-ray spectrometer was carried out by WILLIAMS and JOHNS (7) with the aim of observing low energy internal or external conversion electrons associated with 55.6 and 125 keV transitions in ^{185}Re, if existing. Upper limits of $1.5 \cdot 10^{-3}$ and $9 \cdot 10^{-4}$ were given respectively to branching ratios to the two levels.

All the previous investigations used natural tungsten irradiated with pile neutrons. Recently the decay of ^{185}W was investigated by DUBEY *et al.* (8) and by ARMITAGE and ROSSER (9) by irradiating electromagnetically enriched ^{184}W. By means of a comparison between the γ spectra from natural and enriched tungsten, these authors showed that the 56 keV peak present in the sample of natural tungsten did not occur in the enriched one. No γ-rays were reported by DUBEY *et al.* with upper limits of γ-ray per β-ray ranging between $1.3 \cdot 10^{-3}$ and $3.5 \cdot 10^{-3}$ in the whole spectrum. Furthermore, evidence was found by ARMITAGE and ROSSER of a transition to a level at 125 keV with branching ratio of 10^{-4} or thereabout.

Finally new information on the excited energy levels of ^{185}Re, has become available from the Coulomb excitation studies; several workers (10) have observed rotational levels at 125 and 286 keV.

The greater part of the quoted investigations appeared while our second research on the decay of ^{185}W was in progress. The aim of this investigation, whose results will be given in the following sections, was to re-examine the decay of ^{185}W by using a sample of ^{185}W obtained by irradiation with neutrons of electromagnetically enriched ^{184}W and by measuring the half-life of the γ-rays arising from the active sample. The timeliness of such measurement must be emphasized since various impurities appear to be present in all the samples examined until now. In fact although various techniques were employed in all the previous measurements, accurate determinations of half-life are lacking.

(5) T. C. SEBASTIAN and A. H. WEBER: *Bull. Am. Phys. Soc.*, **1**, 135 (1956).
(6) N. MARTY and M. VERGNES: *Journ. Phys. et Rad.*, **17**, 1013 (1956).
(7) I. R. WILLIAMS and M. W. JOHNS: *Nuovo Cimento*, **6**, 18 (1957).
(8) V. S. DUBEY, C. E. MANDEVILLE, A. MUKERJI and V. R. POTNIS: *Phys. Rev.*, **106**, 785 (1957).
(9) B. H. ARMITAGE and W. G. W. ROSSER: *Proc. Phys. Soc.*, **71**, 355 (1958).
(10) K. ALDER, A. BOHR, B. MOTTELSON and A. WINTHER: *Rev. Mod. Phys.*, **28**, 432 (1956).

2. – Experimental method and results.

An electromagnetically enriched sample of ^{184}W was supplied by the Atomic Energy Research Establishment of Harwell. Their mass analysis of the sample gave the following results: the masses 186, 184, 183, 182 are present respectively with the relative abundances 2.74, 94.35, 2.08, 0.84%; the mass 180 was not detected, its relative abundance was certainly lower than 0.1% and probably less than 0.01%. Their spectrochemical analysis detected the following impurities: Ag, Cu, Na, Si, Sn with relative abundances ranging from 0.002 to 0.05%. The enriched sample of WO_3 was irradiated by thermal neutrons in the Harwell pile (BEPO). The specific activity obtained was 34 mc/g.

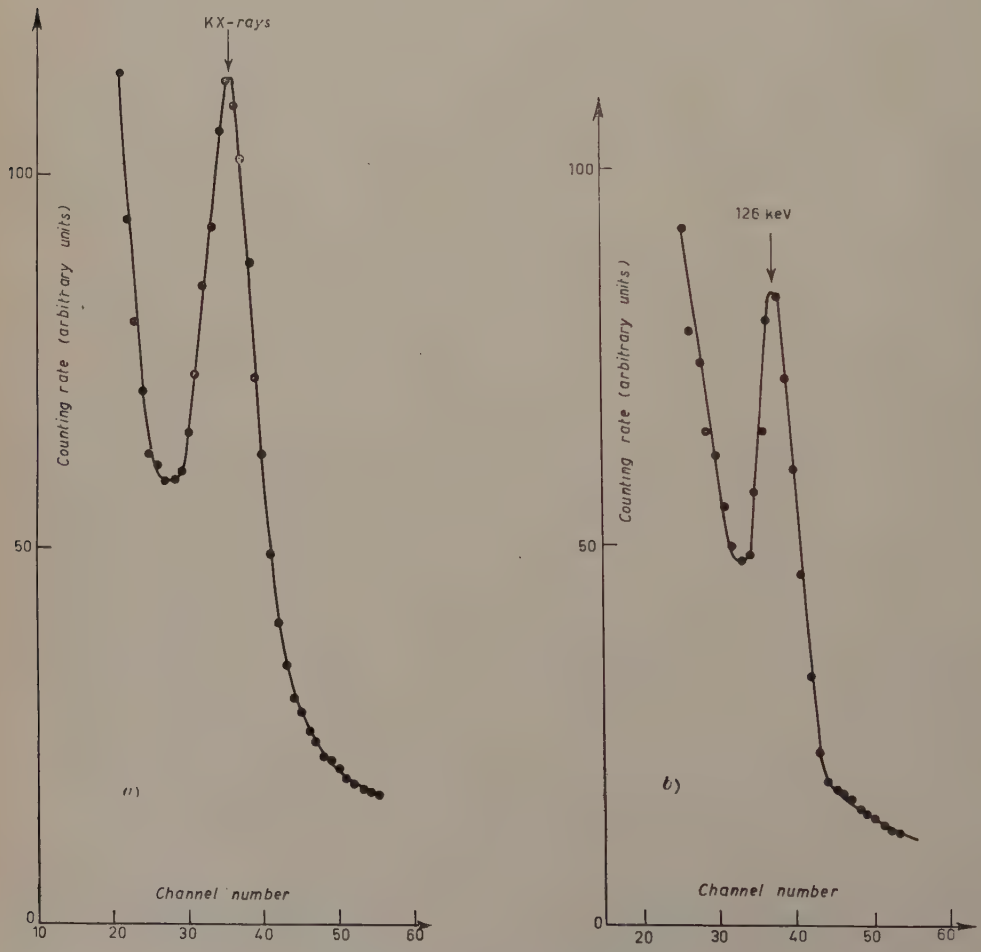

Fig. 1. – *a*) *K*X-rays of Re; *b*) 126 keV γ-rays.

In order to minimize the production of external bremsstrahlung and characteristic K X-rays by β particles on the source holder and on the source itself, thin samples of an average thickness of about 160 $\mu g/cm^2$ were prepared by evaporating a few drops of $(NH_4)_2WO_4$ on a polyethylene foil.

The γ spectra were investigated in a conventional manner, *i.e.* by means of a single crystal, NaI(Tl) ($1\frac{1}{2}$ in. diameter, 1 in. thickness) in connection with a photomultiplier DuMont 6292; the pulse size was measured by means of a ten channel electronic pulse analyzer.

Peaks were observed at the following energies 60, 126, 305, 465, 590, 875 keV. The last three have a much lower intensity than the 60 keV radiation and when observed over a period of several months they appeared to decay with a half-life shorter than the half-life of ^{185}W. The peaks at lower energies 60 and 126 keV are shown in Fig. 1; although superimposed on the continuous bremsstrahlung spectrum they are clearly visible. After calibration of the spectrometer their energy resulted to be:

$$(60.1 \pm 0.4)\ \text{keV},$$

$$(126 \pm 2)\ \text{keV}.$$

Taking into account the energy value and the consideration which will follow, the first peak was interpreted as due to characteristic K X-rays of Re.

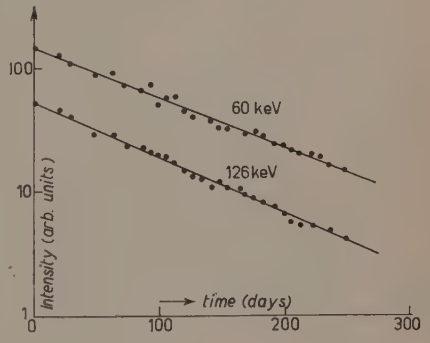

Fig. 2. – Decay curves of the radiations from ^{185}W.

The half-lives of the radiations were measured by observing the time decrease of the area under the peaks over a period of 250 days, that is more than three half-lives of ^{185}W ($T_{\frac{1}{2}} = 73.2$ d). The results are shown in Fig. 2. A mean square adjustement of experimental points gives the values shown in Table I. In the same table we report the intensity of the various radiations per β-ray corrected for any instrumental effects. The β-ray intensity was

TABLE I.

Energy of radiations (keV)	Intensity per β-ray	Half-life (days)
60.1	$1.54 \cdot 10^{-3}$	76.2 ± 2.3
126	$0.56 \cdot 10^{-3}$	69.5 ± 3.5
305	$0.10 \cdot 10^{-3}$	68.8 ± 3.1

determined from the area of an ideal Fermi distribution, by means of a split-crystal spectrometer whose counting efficiency is essentially 100% and using a weaker sample than the one employed for γ-ray spectra.

3. – Concluding remarks.

The half-lives of the radiations quoted in Table I appear at first glance to be the same as that of ^{185}W. In spite of this the attribution of the 305 keV radiation to the decay of ^{185}W is questionable: its half-life, energy and very weak intensity can be accounted by the presence of minute traces ($\sim 10^{-6}$) of ^{191}Ir in the target material. That the 126 keV radiation is due to ^{185}W can nevertheless be considered certain in view of the following facts:

1) No γ-rays known at present have 126 keV energy and decay with a half-life equal to 69.5 d.

2) The radiation energy is in very good agreement with the value obtained for the first rotational level of ^{185}Re ([10]).

3) If it is assumed that the 60 keV peak is due to K X-rays characteristic of Re arising from K internal conversion of 126 keV γ-rays, the following value is obtained for K internal conversion of this γ-ray:

$$\alpha_K = 2.9 \pm 0.3 \,. \tag{1}$$

For the K-shell fluorescence yield of Re the value: $\omega_K = 0.95$ was adopted ([11]).

The value (1) is in reasonable agreement with the theoretical conversion coefficient for an $M1$ transition ($\alpha_K = 2.65$): such type of transition would be expected on the basis of the unified nuclear model of Bohr and Mottelson for the de-excitation of the first excited level of ^{185}Re. Our results can be also compared with the K conversion coefficient obtained by Mc Gowan and Stelson ([12]) by measuring the ratio of K X-ray to γ-ray intensity for a α particle Coulomb excitation of ^{185}Re. The value of Mc Gowan and Stelson is $\alpha_K = 2.10 \pm 0.43$.

On the basis of the preceding considerations the proposed decay scheme is shown in Fig. 3. As can be seen the well known first rotational level of

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([11]) I. Bergström in K. Siegbahn: *Beta and Gamma-Ray Spectroscopy* (Amsterdam, 1955), p. 624.

([12]) F. K. Mc Gowan and P. H. Stelson: *Phys. Rev.*, **103**, 1133 (1956).

^{185}Re is populated by the β-decay of ^{185}W. As regards the branching ratio to the first excited level of ^{185}Re (0.3 %), it was obtained by taking into account the data of Table I, the above-quoted K conversion coefficient and the K/L conversion ratio given by BERNSTEIN and LEWIS ($K/L = 2.2$) ([13]). The last ratio, nevertheless, influences the final result very little.

We are now able to deduce the log ft value for the β-transition to the excited level. It results: log $ft = 9.6$ which corresponds to a first forbidden β transition ($\Delta I = 2$; yes). In connection with this value we observe that the assignement of orbitals to the levels involved in the transition is: 3/2 — to the ground level of ^{185}W, on the basis of nuclear shell model, and 7/2+ to the first excited level of ^{185}Re, because of its rotational character.

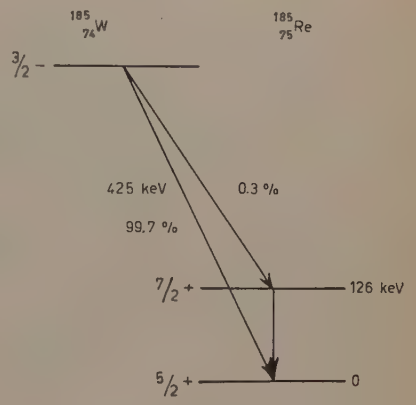

Fig. 3. – Proposed decay scheme of ^{185}W.

Our conclusions on the first excited level of ^{185}Re are in good agreement with those obtained in Coulomb excitation of Re. Agreement is also found with the results of ARMITAGE and ROSSER ([9]) on the decay of ^{185}W, though only on a qualitative basis because of the different values of the branching ratio.

([13]) E. M. BERNSTEIN and H. W. LEWIS: *Bull. Am. Phys. Soc.* II, **2**, 51 (1957).

RIASSUNTO

Si esamina lo spettro γ emesso da un campione di ^{184}W arricchito elettromagneticamente e irraggiato con neutroni termici. Il decadimento delle diverse radiazioni osservate è stato seguito per più di tre periodi di dimezzamento del ^{185}W. Si è accertata l'esistenza di una transizione β al primo livello rotazionale del ^{185}Re di 125 keV. È stato misurato il coefficiente di conversione K della radiazione di diseccitazione e si è trovato: $\alpha_K = 2.9 \pm 0.3$.

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On the Compatibility of Relativistic Wave Equations for Particles of Higher Spin in the Presence of a Gravitational Field.

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(ricevuto il 23 Giugno 1958)

Summary. — It is known that the equations for particles of spin $\geqslant \frac{3}{2}$ become inconsistent when electromagnetic interactions are introduced by the usual rules. An analogous problem is that of the effect of the introduction of gravitational fields on the consistency of the equations. The « natural generalization » of the usual field equations for particles of spin $\frac{3}{2}$ is here considered in detail, and it is shown that the equations are compatible if and only if the Riemann space in which the spinor field is contemplated is an Einstein space. This result is discussed.

1. – Introduction.

Relativistic wave equations for particles of arbitrary spin were proposed by Dirac (1), and later considered in greater detail by Fierz (2). Both authors confined themselves to the case in which interactions with external fields were absent, except insofar as Dirac suggested that electromagnetic interactions might be allowed for by the usual rule of replacing the operator $-i\partial/\partial x^k$ by the operators $-i\partial/\partial x^k - e\varphi_k/hc$, where φ_k is the electromagnetic potential. However, it was pointed out by Fierz and Pauli (3) that for particles of spin $\geqslant \frac{3}{2}$ the resulting field equations are inconsistent. They show that consistency may be restored by deriving the field equations and additional con-

(1) P. A. M. Dirac: *Proc. Roy. Soc.*, A **155**, 447 (1936).
(2) M. Fierz: *Helv. Phys. Acta*, **12**, 3 (1939).
(3) M. Fierz and W. Pauli: *Proc. Roy. Soc.*, A **173**, 211 (1939).

ditions from a variational principle, the Lagrangian being constructed with the aid of certain auxiliary spinors of valence less than that of the main field spinors. This method is however rather artificial and certainly unwiedly.

The question now arises as to the consistency of field equations for particles of higher spin in the presence of a gravitational field. Unlike electromagnetic interaction, gravitational interaction must always be present. If the field quantities be taken as spinors which are symmetrical in their undotted and in their dotted indices, I shall regard as the « natural field equations » for the case in hand those equations which arise from the equations of Dirac and Fierz if the ordinary derivatives are replaced by covariant derivatives, as defined in the calculus of INFELD and VAN DER WAERDEN (4). Since the problem in hand leads to various difficulties in the case of arbitrary spin I shall confine myself to one of the cases treated in detail by FIERZ and PAULI (3), namely to the equations for particles of spin $\frac{3}{2}$. The main result which then emerges is that *the natural field equations for particles of spin* $\frac{3}{2}$ *are consistent if and only if the Riemann space in which they are contemplated is an Einstein space* (*). The implications of this are briefly discussed in the last section.

2. – The equations for spin $\frac{3}{2}$.

The field equations for spin $\frac{3}{2}$ as given by FIERZ and PAULI may be written

$$p^{\lambda}{}_{\nu}\xi^{\dot{\mu}\nu\varrho} + p^{\dot{\mu}}{}_{\nu}\xi^{\lambda\nu\varrho} = 2\varkappa\eta^{\lambda\dot{\mu}\varrho}\,, \tag{1}$$

$$p_{\dot{\lambda}}{}^{\nu}\eta^{\lambda\dot{\mu}\varrho} + p_{\dot{\lambda}}{}^{\varrho}\eta^{\lambda\dot{\mu}\nu} = 2\varkappa\xi^{\dot{\mu}\nu\varrho}\,, \tag{2}$$

$\xi^{\dot{\mu}\nu\varrho}$ and $\eta^{\lambda\dot{\mu}\varrho}$ consequently being symmetric in their dotted and in their undotted indices. So that (1) and (2) shall describe particles of spin $\frac{3}{2}$ only, the following additional conditions must be added

$$p_{\dot{\mu}\nu}\xi^{\dot{\mu}\nu\varrho} = 0\,, \tag{3}$$

$$p_{\dot{\lambda}\varrho}\eta^{\lambda\dot{\mu}\varrho} = 0\,. \tag{4}$$

The operator $p^{\dot{\mu}\nu}$ here means $\sigma^{k\dot{\mu}\nu}(\ )_{,k}$, subscripts following a comma denoting the ordinary (partial) derivative. Because of (3) and (4), equations (1) and (2)

(4) L. INFELD and B. L. VAN DER WAERDEN: *Sitz. Ber. d. Preuss. Akad. d. Wiss.*, **9**, 380 (1933). Their notation will be adhered to throughout.

(*) An Einstein space, *i.e.* a V_4 for which $R_{st} = \lambda g_{st}$, will sometimes be denoted by V_4^{E}.

may be replaced by

$$p^{\dot\lambda}{}_{\alpha}\xi^{\dot\mu\alpha\varrho} = \varkappa\eta^{\dot\lambda\dot\mu\varrho}\,, \tag{5}$$

$$p_{\dot\lambda}{}^{\nu}\eta^{\dot\lambda\dot\mu\varrho} = \varkappa\xi^{\dot\mu\nu\varrho}\,. \tag{6}$$

The presence of a gravitational field is now to be allowed for through forming the natural generalization of equations (3)–(6) by interpreting $p^{\dot\mu\nu}$ to be the operator

$$p^{\dot\mu\nu} = \sigma^{k\dot\mu\nu}(\ )_{;k}\,, \tag{7}$$

where subscripts following a semicolon denote covariant differentiation, and the $\sigma^{k\dot\mu\nu}$ are the basic vector spinors appropriate to the space-time in question.

It should be noted that there is no point in using the gauge-covariant form of the spinor calculus since otherwise one is merely confronted again with the presence of electromagnetic terms which leads to the inconsistencies of the equations above which have already been examined by FIERZ and PAULI. Accordingly one has here

$$\gamma_{\mu\nu;s} = 0\,, \qquad \Gamma^{\lambda}_{\lambda s} = \Gamma^{\dot\lambda}_{\dot\lambda s}\,, \tag{8}$$

and the expression for the spin curvature tensor in terms of the Riemann-Christoffel tensor is

$$P^{\cdot l}{}_{\nu k l} = \tfrac{1}{2}\sigma^{s\dot\lambda\mu}\sigma^{t}{}_{\dot\lambda\nu}R_{\cdot tkl}\,. \tag{9}$$

3. – The condition of consistency.

a) Applying the operator $p_{\dot\lambda}{}^{\nu}$ to both sides of eq. (5) and using (6) one has

$$\sigma^{k\dot\lambda}{}_{\alpha}\sigma^{l}{}_{\dot\lambda}{}^{\nu}\xi^{\dot\mu\alpha\varrho}{}_{;kl} = \varkappa\sigma^{l}{}_{\dot\lambda}{}^{\nu}\eta^{\dot\lambda\dot\mu\varrho} = \varkappa^2\xi^{\dot\mu\nu\varrho}\,. \tag{10}$$

The left hand member of (10) may be transformed as follows:

$$\sigma^{k\dot\lambda}{}_{\alpha}\sigma^{l}{}_{\dot\lambda}{}^{\nu}\xi^{\dot\mu\alpha\varrho}{}_{;kl} = \sigma^{k\dot\lambda}{}_{\alpha}\sigma^{l}{}_{\dot\lambda}{}^{\nu}(\xi^{\dot\mu\alpha\varrho}{}_{;(kl)} + \xi^{\dot\mu\alpha\varrho}{}_{;[kl]}) = -\tfrac{1}{2}\Box\xi^{\dot\mu\nu\varrho} + \sigma^{k\dot\lambda}{}_{\alpha}\sigma^{l}{}_{\dot\lambda}{}^{\nu}\xi^{\dot\mu\alpha\varrho}{}_{;[kl]}\,, \tag{11}$$

since

$$\sigma^{(k}{}_{\dot\lambda\nu}\sigma^{l)\dot\lambda\alpha} = \tfrac{1}{2}g^{kl}\delta^{\alpha}_{\nu}\,. \tag{12}$$

The right hand member of (11) must be symmetric in ν and ϱ. Accordingly,

transvecting it with $\gamma_{\nu\varrho}$ one obtains the equation

$$\sigma^{k\dot{\lambda}}{}_{\alpha}\sigma^{l}{}_{\dot{\lambda}\beta}\xi^{\dot{\mu}\alpha\beta}{}_{;[kl]} = 0 \tag{13}$$

as the necessary and sufficient condition for the consistency of the field equations. It may be written in the form

$$\sigma^{k\dot{\lambda}}{}_{\alpha}\sigma^{l}{}_{\dot{\lambda}\beta}(P^{\dot{\mu}}{}_{\dot{\varrho}kl}\xi^{\dot{\varrho}\alpha\beta} + P^{\varkappa}{}_{\varrho kl}\xi^{\dot{\mu}\alpha\beta} + P^{\beta}{}_{\varrho kl}\xi^{\dot{\mu}\alpha\varrho}) = 0\,. \tag{14}$$

b) Since $\xi^{\dot{\mu}[\alpha\varrho]} = 0$ and $P^{\beta}{}_{\varrho(kl)} = 0$ the third member on the left of (14) is equal to the second. In the following the identity (cf. HARISH-CHANDRA (5))

$$\sigma^{a}{}_{\dot{\alpha}\nu}\sigma^{b\dot{\alpha}\beta}\sigma^{c}{}_{\dot{\mu}\beta} = \tfrac{1}{2}(g^{ab}\sigma^{c}{}_{\dot{\mu}\nu} + g^{bc}\sigma^{a}{}_{\dot{\mu}\nu} - g^{ac}\sigma^{b}{}_{\dot{\mu}\nu} + ie^{abcd}\sigma_{d\dot{\mu}\nu}) \tag{15}$$

will be used repeatedly. Here

$$e^{abcd} = \widetilde{\omega}\varepsilon^{abcd}/\sqrt{-g} \tag{16}$$

is a skew-symmetric tensor. ε^{abcd} is the numerical tensor-density of Levi-Civita, whilst $\widetilde{\omega}$ is a W-scalar (6), whose value, having been prescribed to be $+1$ in some right-handed co-ordinate system, upon transformation to a second co-ordinate system becomes $+1$ or -1 according as the Jacobian of the transformation is positive or negative.

Now, using (9) and (15),

$$\begin{aligned}\sigma^{l}{}_{\dot{\lambda}\beta}P^{\beta}{}_{\varrho kl} &= \tfrac{1}{2}\sigma^{t}{}_{\dot{\alpha}\varrho}\sigma^{s\dot{\alpha}\beta}\sigma^{l}{}_{\dot{\lambda}\beta}R_{stkl} \\ &= \tfrac{1}{4}(g^{ts}\sigma^{l}{}_{\dot{\lambda}\varrho} + g^{sl}\sigma^{t}{}_{\dot{\lambda}\varrho} - g^{tl}\sigma^{s}{}_{\dot{\lambda}\varrho} + ie^{tslm}\sigma_{m\dot{\lambda}\varrho})R_{stkl}\,.\end{aligned} \tag{17}$$

Since, identically,

$$R_{stkl} = R_{klst} \quad \text{and} \quad R_{k[lst]} = 0, \tag{18}$$

one has $e^{tslm}R_{klst} = 0$, so that the last member of (17) becomes $\frac{1}{2}\sigma^{t}{}_{\dot{\lambda}\varrho}R_{tk}$. Hence

$$\sigma^{k\dot{\lambda}}{}_{\alpha}\sigma^{l}{}_{\dot{\lambda}\beta}P^{\beta}{}_{\varrho kl} = \tfrac{1}{2}\sigma^{k\dot{\lambda}}{}_{\alpha}\sigma^{t}{}_{\dot{\lambda}\varrho}R_{tk} = \tfrac{1}{4}\gamma_{\varrho\varkappa}R\,, \tag{19}$$

by (12).

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(5) HARISH-CHANDRA: *Proc. Ind. Acad. Sci.*, **23**, 152 (1946). (See also footnote (6)).

(6) The terminology is that of J. A. SCHOUTEN: *Ricci-Calculus*, Second Edition (Berlin, 1954), Chap. I, p. 12. The factor $\widetilde{\omega}$ is absent from Harish-Chandra's relations. One often finds «identities» in the literature between tensors and W-tensors.

It thus follows that the second and third terms of the left hand member of (14) vanish identically. The first term is more troublesome. Using (9) one has

$$\Omega^{\dot\mu}{}_{\dot\varepsilon}{}^{\nu}{}_{\varrho} \equiv \sigma^{k\dot\lambda\nu}\,\sigma^{l}{}_{\dot\lambda\varrho}P^{\dot\mu}{}_{\dot\varepsilon kl} = \tfrac{1}{2}\sigma^{k\dot\lambda\nu}\sigma^{l}{}_{\dot\lambda\varrho}\sigma^{s\dot\mu\beta}\sigma^{t}{}_{\dot\varepsilon\beta}R_{klst}$$

Multiplying by $\sigma^{a}_{\dot\mu\nu}$ throughout

$$\begin{aligned}(20)\qquad \sigma^{a}_{\dot\mu\nu}\Omega^{\dot\mu}{}_{\dot\varepsilon}{}^{\nu}{}_{\varrho} &= \tfrac{1}{2}\sigma^{k\dot\lambda\nu}\sigma^{a}_{\dot\mu\nu}\sigma^{s\dot\mu\beta}\sigma^{t}{}_{\dot\varepsilon\beta}\sigma^{l}{}_{\dot\lambda\varrho}R_{klst}\\ &= \tfrac{1}{4}\sigma^{k\dot\lambda\nu}\sigma^{l}{}_{\dot\lambda\varrho}(2g^{as}\sigma^{t}{}_{\dot\varepsilon\nu} + ie^{astr}\sigma_{r\dot\varepsilon\nu})R_{klst}\,,\end{aligned}$$

upon transforming the factor consisting of the second, third and fourth σ-symbol by means of (15). The latter may now be applied again to each of the two terms of the last member of (20). Thus, keeping (18) and the skew-symmetry of R_{klst} in k, l and in s, t respectively constantly in mind, one obtains

$$(21)\qquad \sigma^{a}_{\dot\mu\nu}\Omega^{\dot\mu}{}_{\dot\varepsilon}{}^{\nu}{}_{\varrho} = \tfrac{1}{8}(4g^{as}g^{kt}\sigma^{l}{}_{\dot\varepsilon\varrho} - e^{ast}{}_{r}e^{lkrm}\sigma_{m\dot\varepsilon\varrho})R_{klst}\,.$$

Now

$$\begin{aligned}(22)\qquad e^{ast}{}_{r}e^{lkrm}\sigma_{m\dot\varepsilon\varrho}R_{klst} &= e^{astr}e_{lkrm}\sigma^{m}{}_{\dot\varepsilon\varrho}R^{kl}{}_{st}\\ &= -\delta^{astr}_{lkrm}\sigma^{m}{}_{\dot\varepsilon\varrho}R^{kl}{}_{st} = +\delta^{ast}_{lkm}\sigma^{m}{}_{\dot\varepsilon\varrho}R^{kl}{}_{st}\\ &= 2(\delta^{a}_{l}\delta^{s}_{k}\delta^{t}_{m} + \delta^{a}_{k}\delta^{s}_{m}\delta^{t}_{l} + \delta^{a}_{m}\delta^{s}_{l}\delta^{t}_{k})\sigma^{m}{}_{\dot\varepsilon\varrho}R^{kl}{}_{st}\\ &= 2(\delta^{a}_{m}R - 2R^{a}{}_{m})\sigma^{m}{}_{\dot\varepsilon\varrho}\,.\end{aligned}$$

(21) therefore becomes

$$\sigma^{a}_{\dot\mu\nu}\Omega^{\dot\mu}{}_{\dot\varepsilon}{}^{\nu}{}_{\varrho} = (R^{am} - \tfrac{1}{4}g^{am}R)\sigma_{m\dot\varepsilon\varrho}\,.$$

Hence finally

$$(23)\qquad \Omega^{\dot\mu}{}_{\dot\varepsilon}{}^{\nu}{}_{\varrho} = \sigma^{k\dot\mu\nu}\sigma^{l}{}_{\dot\varepsilon\varrho}(R_{kl} - \tfrac{1}{4}g_{kl}R) = \sigma^{k\dot\mu\nu}\sigma^{l}{}_{\dot\varepsilon\varrho}R_{kl} - \tfrac{1}{4}\delta^{\dot\mu}_{\dot\varepsilon}\delta^{\nu}_{\varrho}R\,.$$

Keeping in mind that $\xi^{\dot\mu[\lambda\varrho]} = 0$, equation (14) accordingly reduces now simply to

$$(24)\qquad \sigma^{k\dot\mu}{}_{\alpha}\sigma^{l}{}_{\dot\varrho\beta}R_{kl}\xi^{\dot\varrho\alpha\beta} = 0\,.$$

c) Equation (24) constitutes conditions which must be satisfied for *arbitrary* values of the $\xi^{\dot\varrho\alpha\beta}$. This may for instance be inferred as follows. As usual one may assume on physical grounds that in the region considered co-ordinates can be chosen such that

$$(25)\qquad g_{kl} = \eta_{kl} + h_{kl}\,,$$

where η_{kl} is the galilean metric tensor and h_{kl} is « small ». Then in the first approximation equations (3)–(6) reduce to the corresponding equations in galilean space-time, and one may consider plane wave solutions

$$\xi^{\dot{\mu}\dot{\nu}\dot{\varrho}} = a^{\dot{\mu}\dot{\nu}\dot{\varrho}} \exp\left[i\, 2^{\frac{1}{2}} \varkappa (l_s x^s)\right] \tag{26}$$

of (3), (4), where the $a^{\dot{\mu}\dot{\nu}\dot{\varrho}}$ are arbitrary constants and the l_s are four constants such that

$$\eta^{st} l_s l_t = 0\,. \tag{27}$$

Eqs. (5) and (6) will be satisfied if in addition the $a^{\dot{\mu}\dot{\nu}\dot{\varrho}}$, l_s are subjected to the restriction

$$\sigma^k_{\dot{\mu}\dot{\nu}} a^{\dot{\mu}\dot{\nu}\dot{\varrho}} l_k = 0\,. \tag{28}$$

When the first order approximation (26) is inserted into the second order condition (24) one has algebraic conditions upon the $a^{\dot{\mu}\dot{\nu}\dot{\varrho}}$ into which the l_s do not enter. There will exist solutions for any arbitrary choice of the six constants (amplitudes) $a^{\dot{\mu}\dot{\nu}\dot{\varrho}}$, for (27) and (28) then impose only three restrictions upon the four components of the propagation vector l_s. But all such solutions are subject to (24). This requires that

$$(\sigma^{k\dot{\mu}}{}_{\alpha} \sigma^l_{\dot{\varrho}\beta} + \sigma^{k\dot{\mu}}{}_{\beta} \sigma^l_{\dot{\varrho}\alpha}) R_{kl} = 0\,, \tag{29}$$

or upon multiplication by $-\sigma^s{}_{\dot{\mu}}{}^{\alpha} \sigma^{t\dot{\varrho}\beta}$

$$(g^{ks} g^{lt} + \sigma^s_{\dot{\mu}\alpha} \sigma^{k\dot{\mu}\beta} \sigma^t_{\dot{\varrho}\beta} \sigma^{l\dot{\varrho}\alpha}) R_{kl} = 0\,.$$

Once again applying (15) this becomes

$$[g^{ks} g^{lt} + \tfrac{1}{2}(g^{sk} \sigma^t_{\dot{\varrho}\alpha} + g^{kt} \sigma^s_{\dot{\varrho}\alpha} - g^{st} \sigma^k_{\dot{\varrho}\alpha} + i e^{skt r} \sigma_{r\dot{\varrho}\alpha}) \sigma^{l\dot{\varrho}\alpha}] R_{kl} =$$

$$= [g^{ks} g^{lt} + \tfrac{1}{2}(g^{sk} g^{lt} + g^{kt} g^{sl} - g^{st} g^{kl} + i e^{sktl})] R_{kl} = 0\,,$$

that is,

$$R^{st} - \tfrac{1}{4} g^{st} R = 0\,. \tag{30}$$

In view of the identity of Bianchi (30) is equivalent to

$$R_{st} = \lambda g_{st}\,, \qquad (\lambda = \text{const.}), \tag{31}$$

which expresses the condition that the V_4 be an Einstein space. Conversely, if (31) is satisfied then so is (24) as follows by inspection. Hence one has the result announced at the end of Sect. **1**.

d) Taking (19) and (23) into account, one obtains from (10) and (11), when (31) is satisfied, the second order wave equation for $\xi^{\dot{\mu}\nu\varrho}$, *viz.*

$$\Box\xi^{\dot{\mu}\nu\varrho} + (2\varkappa^2 + \lambda)\xi^{\dot{\mu}\nu\varrho} + \sigma^{k\dot{\lambda}}{}_{\alpha}\sigma^{l}{}_{\lambda}{}^{\nu}P^{\varrho}{}_{\beta kl}\xi^{\dot{\mu}\alpha\beta} = 0\,, \tag{32}$$

which, in contradistinction to the case of Dirac's equations (for particles of spin $\frac{1}{2}$) also contains the *uncontracted* curvature tensor. (32) is of course compatible with the condition $p_{\dot{\mu}\nu}\xi^{\dot{\mu}\nu\varrho} = 0$. (The process of verifying this directly is very tedious. One finds eventually that compatibility is assured if

$$P^{\varrho}{}_{\lambda st} + \tfrac{1}{2}ie^{ab}{}_{st}P^{\varrho}{}_{\lambda ab} = 0\,; \tag{33}$$

and this relation is indeed satisfied in an Einstein space, as follows from (23).)

4. – Conclusion.

As pointed out in the introduction, the equations for higher spin fields become inconsistent when one attempts to introduce electromagnetic interactions in the usual way. It now emerges (for spin $\frac{3}{2}$ at any rate) that in a certain sense electromagnetic interactions will in principle also lead to inconsistency for a different reason, namely because in the presence of an electromagnetic field the V_4 will not be a V_4^{E}. When electromagnetic (and other non-gravitational) fields are absent compatibility of the equations always obtains (see however the remark below), for, in the present context, it would scarcely be meaningful to introduce a phenomenological matter tensor T^{kl}, so that (31) are just the equations of the gravitational field *per se*. In this sense the state of affairs is satisfactory. However, it must be kept in mind that any reaction of the spinor field upon the gravitational field has here been disregarded; that is, the gravitational field has here been looked upon as a given, passive background field. (Whether deliberately or inadvertently this is done also for instance by EDDINGTON [7] in his discussion of the propagation of electromagnetic waves.) In principle this procedure can hardly be regarded as valid; and the V_4 will not be an Einstein space.

No matter what the metric of the V_4 may be one might of course derive alternative field equations, together with the additional conditions, from a

[7] A. S. EDDINGTON: *The Mathematical Theory of Relativity*, Second edition (Cambridge, 1930), Chap. VI, p. 176.

variational principle after the manner of FIERZ and PAULI [3]. For this purpose two additional spin vectors must be introduced. The variational principle yields expressions for these spin vectors containing factors of the type

$$\sigma^{k\dot{\varrho}}{}_{\alpha}\sigma^{l}{}_{\mu\nu}\xi^{\dot{\mu}\nu\alpha}R_{kl}\,, \tag{34}$$

and these of course vanish when (31) holds. However, the field equations appear rather artificial and will not be further considered here.

RIASSUNTO (*)

È noto che le equazioni per le particelle di spin $\geqslant \frac{3}{2}$ diventano inapplicabili quando si introducono interazioni elettromagnetiche con i metodi usuali. Un problema analogo è quello dell'effetto dell'introduzione di campi gravitazionali sull'applicabilità delle equazioni. Si considera qui dettagliatamente la « generalizzazione naturale » delle usuali equazioni del campo per particelle di spin $\frac{3}{2}$ e si dimostra che le equazioni sono compatibili se, e solo se, lo spazio riemanniano in cui si considera il campo spinoriale è uno spazio einsteiniano. Si discute il risultato.

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On a Compound Model of the Pion-Hyperon Interactions.

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Summary. — The pion-hyperon interactions are considered in the light of a compound model in which a hyperon is taken to be a system of a nucleon plus a meson whose only role is to give the hyperon its observed symmetry properties. It is assumed that pions interact with the nucleon in a hyperon as if the nucleon were free, and that pions do not interact appreciably with the new meson. The pion interactions with the Λ and Σ which follow from this model are equivalent to the interactions of the elementary particle theory of Gell-Mann. The pion interactions with the Ξ are not unique without a further restriction on the model. As an example of this non-uniqueness, two different ΞN potentials are obtained, one of which is approximately the same as the ordinary nucleon-nucleon potential, while the other is much weaker at large distances.

1. – Introduction.

This note concerns the interactions of pions with hyperons on the basis of a compound model of the hyperons. This model bears some resemblance to the model of Goldhaber (1) and Christy (2), in which a hyperon is a bound state of a nucleon and $\overline{K}$ meson. It is also related to the theory of Heisenberg and Pauli (3), in which a hyperon is a nucleon plus a «symmetry changing

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(1) M. Goldhaber: *Phys. Rev.*, **101**, 433 (1956).

(2) R. Christy: *Proceedings, Seventh Annual Rochester Conference on High Energy Physics* (Interscience, 1957).

(3) W. Heisenberg and W. Pauli: unpublished manuscript.

pion » $\tilde{\pi}$. In the Heisenberg-Pauli picture, a K meson is also a composite system, being a combination of a π and $\tilde{\pi}$.

In the following considerations, however, we do not necessarily regard a hyperon as an ordinary bound state of a nucleon and another particle. The essential idea of the compound model, as we discuss it here, is that a hyperon obtains its symmetry properties from the symmetry properties of a nucleon plus the symmetry properties of one other particle. This particle which we denote by δ, is assumed to have spin $s = 0$, isotopic spin $T = 1$, strangeness $S = -1$, and either positive or negative parity. The interactions of δ are assumed to conserve parity, strangeness, and isotopic spin. We may identify δ either with the $\overline{\mathrm{K}}$ of Christy or the $\tilde{\pi}$ of Heisenberg and Pauli.

In order to give the compound model predictive power, we make the following assumptions: One, the interaction of a pion with a nucleon is independent of whether the nucleon is free or combined with δ. Two, the interaction of a pion with δ is much weaker than the pion-nucleon interaction.

2. – Form of the pion-hyperon interactions.

We obtain the pion-hyperon (πY) interactions as follows: Since the Λ and Σ have isotopic spin $T = 0$ and $T = 1$ respectively, these hyperons are singlet and triplet isotopic spin states respectively of a nucleon and δ. The Λ and Σ isotopic spin wave function are

$$(1) \qquad \begin{cases} \Lambda = 2^{-\frac{1}{2}}(p\delta^- - n\delta^0)\,, \\ \Sigma^+ = p\delta^0\,, \quad \Sigma^0 = 2^{-\frac{1}{2}}(p\delta^- + n\delta^0)\,, \quad \Sigma^- = n\delta^-\,. \end{cases}$$

There does not seem to be any way of determining the parity of δ on the basis of this model. Equivalently, the parity of the Λ and Σ relative to the parity of the nucleon cannot be determined. The important point is that the Λ and Σ automatically have the same parity, independently of the parity of δ.

To obtain the πY interactions, we extend the πN interaction to include all baryons B. We write for the πB Hamiltonian

$$(2) \qquad H_\pi = g \sum_{\mathrm{B}} \mathrm{B}\boldsymbol{\tau}\cdot\boldsymbol{\pi}\mathrm{B}\,,$$

where $\boldsymbol{\tau}$ is the usual Pauli isotopic spin operator and we include only the isotopic spin dependence of the interaction. Our notation is that of Gell-Mann [4] in which the symbol for a particle denotes the field operator that annihilates it. To give meaning to $H_{\pi\mathrm{B}}$ when $\mathrm{B} \neq \mathrm{N}$, we let $\boldsymbol{\tau}$ operate on the

[4] M. Gell-Mann: *Phys. Rev.*, **106**, 1296 (1957).

nucleon part of the Λ and Σ wave functions given in (1), with δ playing essentially a geometrical role. The validity of this procedure follows from the assumptions of the previous section. We obtain

$$H_{\pi\Lambda} = g\left[\bar{\Lambda}(\Sigma^0\pi^0 - \Sigma^+\pi^- + \Sigma^-\pi^+) + \text{Herm. conj.}\right], \tag{3}$$

$$H_{\pi\Sigma} = g\left[(\bar{\Sigma}^+\Sigma^+ - \Sigma^-\Sigma^-)\pi^0 + (\bar{\Sigma}^0\Sigma^+ + \bar{\Sigma}^-\Sigma^0)\pi^- + (\bar{\Sigma}^-\Sigma^0 + \bar{\Sigma}^0\Sigma^-)\pi^+\right]. \tag{4}$$

The forms of $H_{\pi\Lambda}$ and $H_{\pi\Sigma}$ in isotopic spin space are specified by charge independence and the requirement that the Hamiltonians be linear in the pion field and quadratic in the baryon fields. We have previously used these forms to calculate ΛN and ΣN potentials and scattering [5]. These Hamiltonians differ from those of GELL-MANN [3] in the relative signs of certain terms, but these sign differences do not have any physical consequences, provided other interactions of the Λ and Σ are neglected.

On the other hand, the forms of $H_{\pi\Lambda}$ and $H_{\pi\Sigma}$ in ordinary space and the strength of the coupling governing the interactions are not specified without more detailed assumptions. The use of the compound model leads to Hamiltonians $H_{\pi\Lambda}$ and $H_{\pi\Sigma}$ which have the same forms in ordinary space and which have the same coupling strength as the Hamiltonians of the elementary particle theory of Gell-Mann [4,6]. Thus as far as the π interactions with Λ and Σ are concerned, the compound model can be regarded as merely a formal device which leads to Hamiltonians which are equivalent to the Gell-Mann Hamiltonians [5]. If we interchange the definitions of Λ and Σ^0 in (1), the Hamiltonians (3) and (4) become identical to those of GELL-MANN.

The situation is different with respect to the Ξ, however. In the compound model, the Ξ is a nucleon and two δ-particles. These two δ's can combine to form a $T=0$ or $T=1$ state, denoted by η_0 and η_1^m respectively. Using η_0 and η_1^m, we can form two linearly independent representations for a $T=\frac{1}{2}$, $S=-2$ particle, which we call Ξ_1, and Ξ_2:

$$\begin{cases} \Xi_1^{\prime} = p\eta_0\,, \qquad \Xi_1^- = n\eta_0\,, \\ \Xi_2^0 = -(1/3)^{\frac{1}{2}} p\eta_1^0 + (2/3)^{\frac{1}{2}} n\eta_1^1\,, \qquad \Xi_2^- = (1/3)^{\frac{1}{2}}\eta_1^0 - (2/3)^{\frac{1}{2}} p\eta_1^{-1}\,. \end{cases} \tag{5}$$

Two other representations for the Ξ,

$$\Xi_3^- = \Lambda\delta^0\,, \qquad \Xi_3^- = \Lambda\delta^-$$

and

$$\Xi_4^0 = -(1/3)^{\frac{1}{2}}\,\Sigma^0\delta^0 + (2/3)^{\frac{1}{2}}\,\Sigma^+\delta^-, \qquad \Xi_4^- = (1/3)^{\frac{1}{2}}\,\Sigma^0\delta^- - (2/3)^{\frac{1}{2}}\,\Sigma^-\delta^0$$

(5) D. LICHTENBERG and M. ROSS: *Phys. Rev.*, **107**, 1714 (1957).
(3) See also J. SCHWINGER: *Annals of Physics*, **2**, 407 (1957).

are not independent of the representations (5). The $\pi\Xi$ interaction is now not unique, but depends on the representation of the Ξ. We consider only the special cases $\Xi = \Xi_1$ and $\Xi = \Xi_2$. For $\Xi = \Xi_1$, the interaction is

$$H_{\pi\Xi_1} = g\left[(\bar{\Xi}^0\Xi^0 - \bar{\Xi}^-\Xi^-)\pi^0 + \sqrt{2}(\bar{\Xi}^0\Xi^-\pi^+ + \bar{\Xi}^-\Xi^0\pi^-)\right], \tag{6}$$

which is the same as that given by GELL-MANN. For $\Xi = \Xi_2$, the interaction is

$$H_{\pi\Xi_2} = -(1/3)g\left[(\bar{\Xi}^0\Xi^0 - \bar{\Xi}^-\Xi^-)\pi^0 + \sqrt{2}(\bar{\Xi}^0\Xi^-\pi^+ + \bar{\Xi}^-\Xi^0\pi^-)\right], \tag{7a}$$

$$H_{\pi\Omega} = (2/3)g\left[\bar{\Xi}^0(\sqrt{3}\,\Omega^+\pi^- - \sqrt{2}\,\Omega^0\pi^0 - \Omega^-\pi^+) + \right. \tag{7b}$$

$$\left. + \bar{\Xi}^-(\Omega^0\pi^- - \sqrt{2}\,\Omega^-\pi^0 - \sqrt{3}\,\Omega^{--}\pi^+) + \text{Herm. conj.}\right],$$

where Ω is a quartet of particles with $S = -2$, which is a combination of $N\delta\delta$ in a $T = 3/2$ configuration. Both the Ξ and Ω have the same parity as the nucleon, independently of the parity of δ. If Ω is a particle, it has not yet been seen. Moreover, one of its states is doubly charged, in contrast to all the known hyperons. It may therefore be preferable to interpret Ω as an unbound system, which may appear as a nucleon and two $\overline{K}$ mesons.

We do not give the baryon interactions with K mesons on the basis of the compound model since we do not see any intuitively obvious way of obtaining these interactions. As one possibility, we might assume, following CHRITSY, that the hyperons are bound states of N and $\overline{K}$, and then use the energies of these states to obtain a potential between N and $\overline{K}$.

On the other hand, the natural way to obtain a field theory of K interactions is to write down the simplest interactions which are linear in the K field and bilinear in the baryon fields. If we demand that the interactions conserve strangeness and isotopic spin, we obtain at once the Gell-Mann interactions. Such a procedure makes no use of the compoound model. In this connection, however, the compound model in the Heisenberg-Pauli form can predict that the K interactions are pseudoscalar. This is because the K is a compoound of π and $\bar{\delta}$, while the Λ and Σ are compounds of N and δ. Since the interactions are linear in the K field and linear in either the Λ or Σ fields, only an even number of δ's occur in the interaction. Thus, the parity of the interaction is the same as the parity of the πN interaction, independently of the parity of δ.

In these considerations, we always assume that the Ξ^0 exists. As it has not yet been seen, an alternative possibility is to assume that only the Ξ^- exists and is an $S = -3$ particle. In this case, the decay $\Xi^- \to \Lambda + \pi^-$ would violate the selection rule $\Delta S = \pm 1$. $H_{\pi\Xi}$, in either the compound model or the Gell-Mann theory, would involve the coupling of the Ξ either to doubly charged particles or to an unbound system of baryons and $\overline{K}$-mesons. We do not discuss this possibility further.

3. – The potential between Ξ and nucleon.

As an example of the different consequences of the interactions (6) and (7), we consider the ΞN potential at low energy. (We do not treat the possibly simpler problem of πΞ scattering as it is much more remote from experiment.)

We first consider the Hamiltonian (6) and make the following approximations:

a) we neglect all baryon mass differences,

b) we neglect the effect of K-mesons, and

c) we neglect Coulomb forces.

The trouble with approximations *a*) and *b*) is that it is difficult to estimate the errors associated with them (7). The advantage is that with these approximations the ΞN potential is identical to the NN potential, which can be taken from experiment. The ΞN scattering cross-sections are different, however, because of the absence of the Pauli principle. Thus, for example, the cross-sections $\sigma(\Xi^-\mathrm{p} \to \Xi^-\mathrm{p})$ and $\sigma(\Xi^-\mathrm{p} \to \Xi^0\mathrm{n})$ can be measured separately, whereas the cross-sections $\sigma(\mathrm{np} \to \mathrm{np})$ and $\sigma(\mathrm{np} \to \mathrm{pn})$ are indistinguisable experimentally. The ΞN cross-section can be given in terms of (spin dependent) scattering amplitudes a_1 and a_0 in the $T=1$ and $T=0$ isotopic spin states respectively,

$$(8)\qquad \begin{cases} \sigma(\Xi^0\mathrm{p} \to \Xi^0\mathrm{p}) = \sigma(\Xi^-\mathrm{n} \to \Xi^-\mathrm{n}) = |a_1|^2, \\ \sigma(\Xi^-\mathrm{p} \to \Xi^-\mathrm{p}) = \sigma(\Xi^0\,\mathrm{n} \to \Xi^0\mathrm{n}) = (1/4)\,|a_1 + a_0|^2, \\ \sigma(\Xi^-\mathrm{p} \to \Xi^0\mathrm{n}) = (1/4)\,|a_1 - a_0|^2. \end{cases}$$

The amplitudes a_1 and a_0 can be directly calculated by solving the two-body Schrödinger equation with the NN potentials.

With the approximations *a*) and *b*) a bound ΞN state, very similar to the deuteron, should occur. It would be electrically neutral and would be a mixture of $\Xi^-\mathrm{p}$ and $\Xi^0\mathrm{n}$ in a $T=0$ state. However, this state would decay in $\sim 10^{-23}$ s via K-meson interactions through the process

$$(10)\qquad \Xi^- + \mathrm{p} \to \Lambda + \Lambda\,.$$

(7) The neglect of the mass difference between Ξ and N means neglecting effects of the order of $(M_\Xi - M_\mathrm{N})/M_\Xi$ or 30%. The error due to the neglect of K-mesons depends on the unknown K coupling constants. For a brief discussion of some effects due to K-mesons and baryon mass differences, see D. LICHTENBERG and M. ROSS: *Proceedings, Padua-Venice Conference on Mesons and Recently Discovered Particles* (1957) and *Phys. Rev.*, **109**, 2163 (1957).

Processes such as (10) also affect the cross-sections (8). However, since the reaction (10) is a K-meson effect, the range of the interaction should be small (of the order of the K Compton wave length). Therefore, at low energy, processes such as (10) should not have a large effect on the elastic scattering cross-sections. The question of whether a bound state of Ξ^-n (or Ξ^0p) should exist is harder to answer. With approximations *a*), *b*) and *c*) the Ξ^-n potential in the singlet state is equal to the pp singlet even-state potential. If we now remove the approximation that the Ξ^- and N have the same mass, keeping the potential fixed, we find that a bound Ξ^- n state exists because of the heavy mass of the Ξ^-. The motivation behind this procedure is that perhaps the pion potentials are not sensitive to mass differences of the baryons. But we do not attach much significance to a conclusion which depends critically on such a hybrid approximation.

We now turn to the evaluation of the Ξ_2N potential. In this case, in addition to using approximations *a*), *b*) and *c*), we need to make an assumption about the nature of Ω. If Ω exists as another baryon, then the interaction (7*b*) must be considered in evaluating the pion contribution to Ξ_2N forces. If the Ω system does not exist as another baryon, then we can, for simplicity, lump Ξ_2-π processes leading to $T = 3/2$ systems (*e.g.* $H_{\pi\Omega}$) with other direct baryon-K meson processes. We shall assume that Ω is not a baryon, and does not play a role. If Ξ contains and admixture of Ξ_2, $H_{\pi\Xi}$ will have the same form as $H_{\pi\Xi_1}$ except that it will have an effective coupling constant $g' \neq g$. For the special case $\Xi = \Xi_2$, $g' = -g/3$. Because of the difference in coupling constants, we cannot give the Ξ_2N potential in terms of the experimental NN potential. We can, however, use any of the calculational methods which numerous authors have used to evaluate the NN potential (8).

In the NN case, rather good agreement with experiment for large NN distances has been obtained by keeping only the second and fourth order terms in an expansion in powers of g. In the Ξ_2N case, the second order term has the same functional form as the second order NN potential, but is of opposite sign and magnitude 1/3 as great. The fourth order Ξ_2N potential has the same sign as the fourth order NN potential and magnitude 1/9 as large, etc. The expansion converges much faster here than in the NN case. However, for small enough distances, the expansion must fail to converge. Also at small distances K-meson effects become important. We can say nothing about the potential at these distances ($\lesssim 0.5 \cdot 10^{-13}$ cm); there is no reason to expect a repulsion, for example. However, at large distances ($\gtrsim 10^{-13}$ cm), the Ξ_2N potential is much weaker than the NN potential. Therefore, either the potential is weak, or, if it is strong, it is confined essentially to the region of the repulsive core of the NN potential. Effects of K-mesons and baryons mass

(8) As one example, see K. Brueckner and K. Watson: *Phys. Rev.*, **92**, 1023 (1953).

differences should not alter this conclusion. We have discussed the special case $\Xi = \Xi_2$ only to show the wide variation in the ΞN potential which can result from the compound model. Some additional restriction is needed to make $H_{\pi\Xi}$ unique.

4. – Conclusions.

We have seen that as far as pion interaction with Λ and Σ are concerned, the compound model is equivalent to the theory of Gell-Mann. The model does not lead to a unique $\pi\Xi$ interaction, but it includes the Gell-Mann interaction as a special case. This case is especially attractive, as it is the only one in which states of Ω do not appear. Using this interaction, the ΞN scattering cross-sections can be obtained independently of perturbation theory, assuming only that effects of K-mesons and baryon mass differences are small. It remains to be seen whether these ΞN scattering cross-sections will be confirmed by experiment.

* * *

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RIASSUNTO (*)

Si esaminano le interazioni pione-iperone nel quadro di un modello composto nel quale un iperone si considera come un sistema di un nucleone più un mesone il cui solo compito è di conferire all'iperone le proprietà di simmetria osservate. Si assume che i pioni interagiscano col nucleone in un iperone come se il nucleone fosse libero, e che i pioni non interagiscano apprezzabilmente col nuovo mesone. Le interazioni dei pioni coi Λ e Σ che derivano da questo modello sono equivalenti alle interazioni della teoria delle particelle elementari di Gell-Mann. Le interazioni dei pioni con i Ξ non risultano univoche se non si impongono al modello ulteriori restrizioni. Come esempio di questa mancanza di univocità, si ottengono due potenziali ΞN, uno dei quali è approssimativamente uguale all'ordinario potenziale nucleone-nucleone, mentre l'altro è molto più debole alle grandi distanze.

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Ultrasonic Absorption in Some Homologous Series of Organic Liquids.

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Summary. — Part I. Results of ultrasonic absorption and velocity measurements in homologous series of fatty acids and alkyl acetates conducted over a range of frequency and temperature are reported. It is shown that the magnitude of the observed absorption and the relaxation frequency bear a relationship to the molecular weight in the series. It is also shown that the relaxation frequency in esters is the same for a common carboxyl group but depends on the molecular weight in a series of esters with different carboxyl radicals. Part II. Values of absorption coefficient and velocity in a number of saturated hydrocarbons — 1-olefines and aromatic hydrocarbons are reported. The relationship between the molecular weight on the one hand and the absorption coefficient and velocity together with their temperature coefficients on the other are discussed for the three series. It is suggested that while in the saturated and unsaturated aliphatic hydrocarbons the excess absorption is due partly to thermal relaxation and partly to bulk viscosity, in the case of aromatic hydrocarbons it is mainly due to thermal relaxation.

Part I. – Acids and Esters.

1. – Introduction.

It is well known that the absorption of ultrasonic waves in liquids is not explained by the classical theory of Stokes-Kirchhoff and various other theories have been proposed to explain the same, *e.g.* thermal relaxation, structural relaxation, bulk viscosity, etc. The existing data on absorption show that no single theory uniquely explains the excessive absorption and the liquids have

been divided up into different groups according to the nature of the liquids and their absorption (1). It has however been found that there are some exceptions to this grouping also (2). It was shown earlier (3) that results of absorption in liquids forming a homologous series would throw some light on the nature of absorption. Data have now been obtained for a few series and two of them consisting of the fatty acids and the acetates are discussed here.

2. – Acids.

Among the earlier work on acids may be mentioned the work of BAZULIN (4-6) in formic acid and acetic acid. This has been entirely confirmed by the later exhaustive work of LAMB and PINKERTON (7) in acetic acid and by the work of KARPOVIČ (8,9) in formic acid. Data by LAMB and HUDDART (10) are also available for propionic acid. Among the many liquids studied so far, these three are the only ones which have shown a positive evidence for relaxation by a change of absorption with frequency. It was therefore considered to be of interest to study other liquids in the fatty acid series. The present paper reports the results of investigation on all the liquid members of the fatty acid series.

A modification (11) of the diffraction method was used in making the observations werein the sound intensity is adjusted for extinction of the zero order. By repeating this setting at different levels and plotting the log of R.F. voltage on the crystal against the distance a straight line is obtained, the slope of which gives α, the absorption constant. The measurements were carried out at room temperature only (~ 30 °C) at four frequencies *viz.* 1.4, 4.3, 7.2 and 15.8 MHz. It is well known that the absorption varies with frequency according to the relation

$$\frac{\alpha}{\nu^2} = B + \frac{A}{1 + (\nu/\nu_m)^2}, \tag{1}$$

(1) J. M. M. PINKERTON: *Proc. Phys. Soc.*, B **62**, 129 (1949).

(2) S. PARHTASARATHY, M. PANCHOLY and A. F. CHHAPGAR: *Temperature Coefficient of Ultrasonic Absorption in Organic Liquids* (under publication in *Nature*).

(3) S. PARTHASARATHY and A. F. CHHAPGAR: *Ann. d. Phys.*, **16**, 297 (1955).

(4) P. BAZULIN: *Phys. Zeits. Sowiet*, **8**, 354 (1935).

(5) P. BAZULIN: *Žu. Èksp. Teor. Fiz.*, **8**, 457 (1938).

(6) P. BAZULIN and J. M. MERSON: *Compt. Rend. URSS*, **24**, 690 (1939).

(7) J. LAMB and J. M. M. PINKERTON: *Proc. Roy. Soc.*, A **199**, 114 (1949).

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(10) J. LAMB and D. H. A. HUDDART: *Trans. Farad. Soc.*, **46**, 540 (1950).

(11) S. PARTHASARATHY, C. B. TIPNIS and M. PANCHOLY: *Zeits. f. Phys.*, **140**, 504 (1955).

where A and B are constants for the liquid corresponding to the maximum and minimum absorptions and ν_m is the relaxation frequency. Substituting the measured values of absorption in this formula, the constants were evaluated and the full curve drawn. The values of these constants obtained thus are in agreement with those obtained by the other workers mentioned earlier and this is taken as a measure of the accuracy of the present work. The velocities were calculated by measuring the line-separation of the diffraction pattern.

Table I shows the results obtained for six acids in the series.

TABLE I.

Liquid	Velocity m/s	$(\alpha/\nu^2)\cdot 10^{17}$ 1.4 MHz	4.3 MHz	7.2 MHz	15.8 MHz	Viscous absorption $\cdot 10^{17}$	$A\cdot 10^{17}$	$B\cdot 10^{17}$	$\alpha\lambda$	ν_m MHz
1. Formic acid	1257	17010	2440	960	280	16	70000	100	0.035	0.80
2. Acetic acid	1126	32780	4900	1870	470	18	112000	100	0.057	0.91
3. Propionic acid	1142	15080	3370	1370	370	17	26600	100	0.025	1.61
4. n-Butyric acid	1178	4760	1740	810	260	23	6000	100	0.010	2.64
5. n-Caproic acid	1250	1000	730	490	220	38	950	100	0.005	5.98
6. n-Caprylic acid	1298	550	480	380	210	63	460	100	0.004	9.01

Fig. 1 shows the variation of velocity with molecular weight in the series. This relationship has been well discussed by various authors [12-14] and hence will not be gone into here. The liquids showed no measurable dispersion of velocity. Fig. 2 shows the variation of α/ν^2 with frequency as calculated from the observations. Fig. 3 shows the variation of α/ν^2 with molecular weight. Because of the relaxation effect there are separate curves for each frequency, but it is seen that at each frequency there is a definite relationship between absorption and molecular weight. The parameters A and B calculated from the above relation (1), which correspond to the values of absorption at $\nu \ll \nu_m$ and $\nu \gg \nu_m$ are also shown, together with the viscous absorption as given by

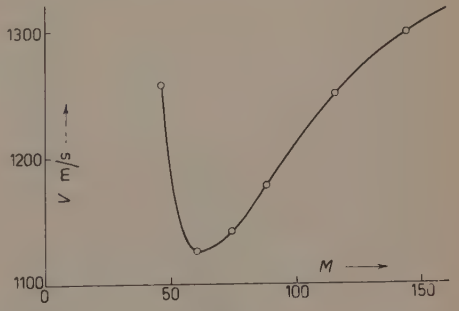

Fig. 1.

[12] S. PARTHASARATHY and N. N. BAKHSHI: *Journ. Sci. Ind. Res.*, A **12**, 448 (1953).
[13] M. R. RAO: *Journ. Chem. Phys.*, **9**, 682 (1941).
[14] W. SCHAAFFS: *Ergeb. Exakt. Naturwiss.*, **25**, 109 (1951).

Stokes' theory. These also show a similar variation with molecular weight. It is also seen that with increasing molecular weight the observed absorption

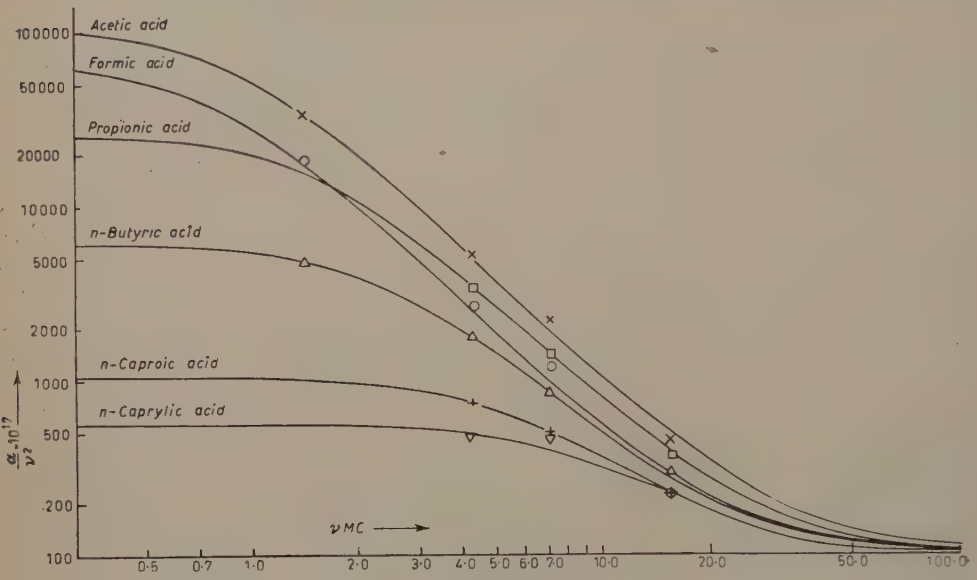

Fig. 2.

tends towards the Stokes value at all frequencies. In Fig. 4, the relaxation frequencies calculated from the data are plotted against molecular weight. The relaxation frequency increases as the molecular weight increases, whereas the observed absorption decreases.

Although observations were taken only at four frequencies and the relaxation frequencies calculated therefrom, it is found that the values agree

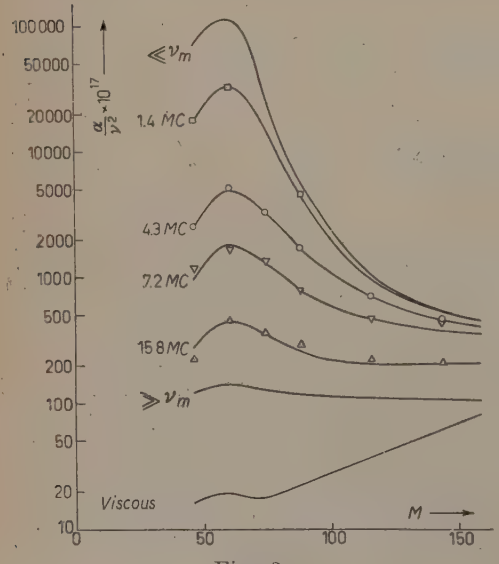

Fig. 3.

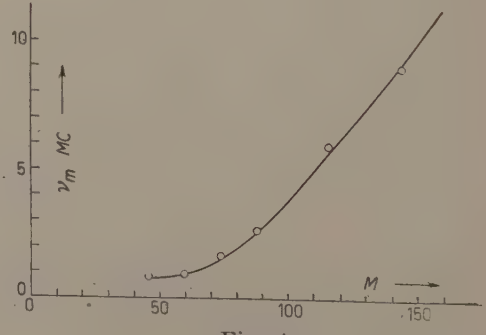

Fig. 4.

with those obtained by other workers wherever the work overlaps. In propionic acid, there is a small disagreement with the values of LAMB and HUDDART. Although these values may be improved upon by a few percent by more extensive measurements, they are nevertheless sufficient to show the trend of relationship existing between absorption and molecular constitution. LAMB and ANDRAE [15] attribute the relaxation in acetic and propionic acids to the excitation of the bending vibration of the carbon-hydrogen bond. PINKERTON [16] on the other hand ascribes the relaxation to the dimerization of one of the hydrogen bonds. Since either of these factors depend upon the molecular weight, the cause of relaxation is similar in all these acids.

3. – Acetates.

The liquids in this series, in particular ethyl acetate, have been studied by many workers. Among these may be mentioned BIQUARD [17], PINKERTON [16], KARPOVICH [9], BÄR [18], PARTHASARATHY, TIPNIS and PANCHOLY [11], VERMA [19] and others. From the available data on ethyl acetate KNESER [20] suggested the existence of two relaxation frequencies at 1 MHz and 70 MHz. In a previous paper [11], the variation of absorption with frequency in the acetates had been studied. In the present paper, the temperature dependence of absorption at one frequency only is reported.

The pulse method was used for the present work. The results at 21 MHz are shown in Table II.

TABLE II.

No.	Liquid	$(\alpha/\nu^2)\cdot 10^{17}$			
		20 °C	30 °C	40 °C	50 °C
1	Methyl acetate . . .	71.2	81.3	86.9	93.0
2	Ethyl acetate	45.2	52.2	57.3	61.2
3	Propyl acetate . . .	64.1	72.8	80.7	87.3
4	Butyl acetate	59.6	62.6	67.4	75.7
5	Amyl acetate	86.0	82.9	79.8	78.2

(15) J. LAMB and J. H. ANDRAE: *Coll. over Ultrasonore Trilligen* (Brussel, 1951), p. 48.
(16) J. M. M. PINKERTON: *Coll. over Ultrasonore Trilligen* (Brussel, 1951), p. 117.
(17) P. BIQUARD: *Ann. d. Phys.*, **6**, 195 (1936).
(18) R. BÄR: *Helv. Phys. Acta*, **10**, 332 (1937).
(19) G. S. VERMA: *Journ. Chem. Phys.*, **18**, 1352 (1950).
(20) H. O. KNESER: *Ergeb. Exakt. Naturwiss.*, **22**, 121 (1949).

It is seen from the results that whereas the temperature coefficient is positive in the working range for the first four members of the series, it becomes negative for amyl acetate. Further, there is no regular sequence in the series as the molecular weight increases. In order to find if the absorption is regular in the series as in the case of the acids, the available data have been collected together and shown in Fig. 5 to 9. - For ethyl acetate, the results of many workers are available, most of these however being at frequencies above 10 MHz, where the results agree among themselves. On the other hand, below 10 MHz the results are fewer and tend to vary. For example, at 1 MHz, VERMA [19] gets a value of $500 \cdot 10^{-17}$ wheareas KARPOVIČ [9] gives the value as $100 \cdot 10^{-17}$. The majority of points however show the pro-

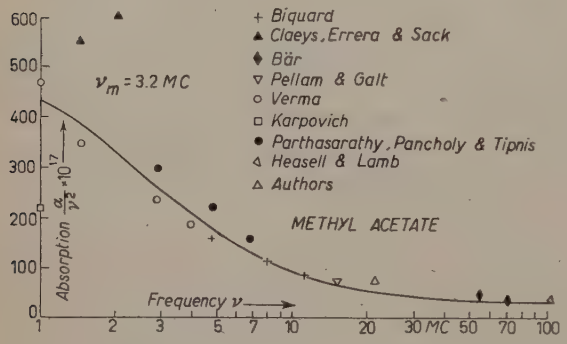

Fig. 5.

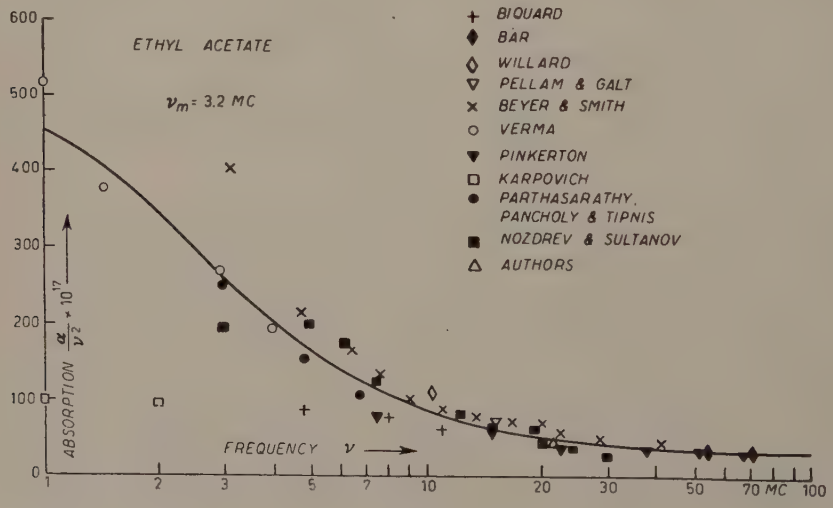

Fig. 6.

bable value to be between these two figures and a mean curve has been drawn to fit the data. For methyl acetate, although not as many results are available, it is again seen that at higher frequencies the results agree better than at lower frequencies. The same is true of the other acetates also, although enough data do not exist. The values of relaxation frequency,

calculated for each liquid, come to about 3 MHz for each liquid, there being no change due to the alkyl radical as in the case of acids. Similarly the maximum and minimum values of absorption are $\alpha/\nu^2 = 500 \cdot 10^{-17}$ and $\alpha/\nu^2 = 50 \cdot 10^{-17}$ respectively for each liquid. It would thus appear that the absorption in acetates is determined by relaxation due to the acetate radical only and the carboxyl radical does not affect it to a very great extent. In order to compare this with other esters like formates, propionates, etc., it is necessary to have complete data for these esters. The only other esters for which data exist are methyl and ethyl formates given by KARPOVIČ ([9]) and by LIEBERMANN ([21]). From these it is seen that the maximum absorption is very much higher in formates than in acetates and the relaxation frequency in the former is of the order of 0.5 MHz for both liquids. Thus it is again seen that the relaxation is determined by the formate radical only.

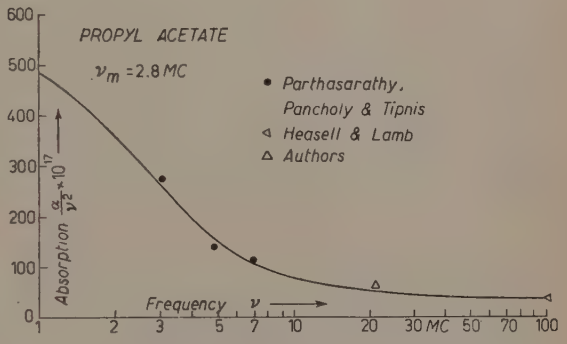

Fig. 7.

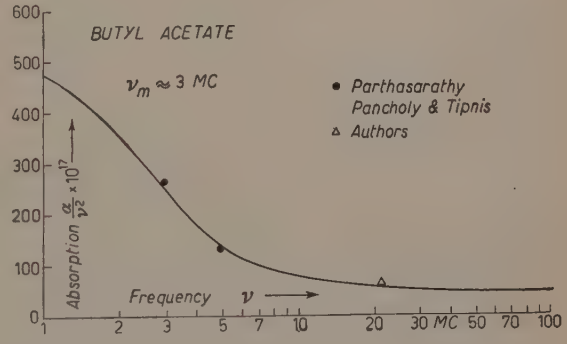

Fig. 8.

It would thus appear that no relation with molecular constitution exists within one set of esters like formates, acetates, etc., but that a definite relation, similar to the acid series, is possible in the different esters of each carboxyl group like methyl formate, methyl acetate, methyl propionate, etc. However, more data

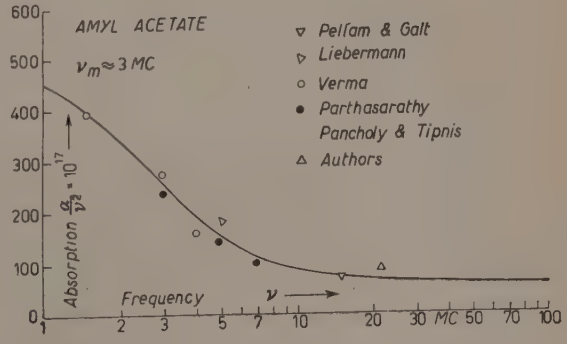

Fig. 9.

([21]) L. N. LIEBERMANN: *Phys. Rev.*, **74**, 1451 (1949).

are required for all these liquids where no data exist at all, as well as more reliable data for the lower frequencies (from 100 kHz to 10 MHz) in those liquids where data are already available.

The following points emerge from the above study:

1) The observed excess absorption in the fatty acids is frequency dependent and of relaxational origin.

2) The observed absorption decreases with increasing molecular weight tending to Stokes' value at very high molecular weights.

3) The relaxation frequency increases regularly with molecular weight.

4) All the acetates examined show the same relaxation frequency while the formates show a higher relaxation frequency. This observation leads to the assumption that in esters the relaxation frequency depends not on the alkyl group but on the carboxyl group.

PART II. - Hydrocarbons.

It has been shown in a previous paper ([2]) that ultrasonic absorption bears a close relation to molecular constitution. Two homologous series—acids and acetates—have already been dealt with earlier ([22]). In the case of acids, the absorption was shown to decrease progressively with the molecular weight. The present paper deals with the series of saturated and unsaturated aliphatic hydrocarbons. For comparison, the series of aromatic hydrocarbons are also discussed.

The measurements were carried out by the pulse method at a frequency of 21 MHz.

Table III gives the velocity and absorption in liquids as a function of temperature of the saturated hydrocarbon series. Fig. 10 shows the temperature variation of velocity. Fig. 11 shows the relationship between velocity at 20 °C and molecular weight. This relationship has already been explained by various

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([22]) S. PARTHASARATHY and A. F. CHHAPGAR: *Ann. d. Phys.*, **16**, 297 (1955).
([23]) See part I of this paper.

TABLE III. – *Aliphatic Hydrocarbons.*

Liquid	Vel. at 20°C m/s	Temp. Coeff. of vel.	Viscous absorpt. at 20 °C $\cdot 10^{17}$	Observed absorption $\cdot 10^{17}$					$\frac{1}{\alpha}\frac{d\alpha}{dT}$
				20°	30°	40°	50°	60 °C	
n-Hexane	1117	— 4.4	8.9	57.6	62.8	68.2	73.8	78.5	+ 0.0084
n-Octane	1199	— 4.1	12.0	49.5	51.0	52.9	54.5	56.0	+ 0.0031
n-Decane	1257	— 3.9	16.3	56.7	56.7	56.7	56.7	56.7	0.0000
n-Dodecane	1306	— 3.8	23.8	69.7	66.3	63.0	60.4	58.5	— 0.0040
n-Tetradecane	1337	— 3.7	30.6	87.0	81.4	76.1	71.0	66.5	— 0.0067
n-Hexadecane	1358	— 3.5	44.8	101.0	93.0	85.9	79.0	72.9	— 0.0081

workers ([24-26]). Fig. 11 also shows the relation between the temperature coefficient of velocity and molecular weight. Thus it is seen that both velocity and its temperature coefficient in the liquids of this series increase with molecular weight.

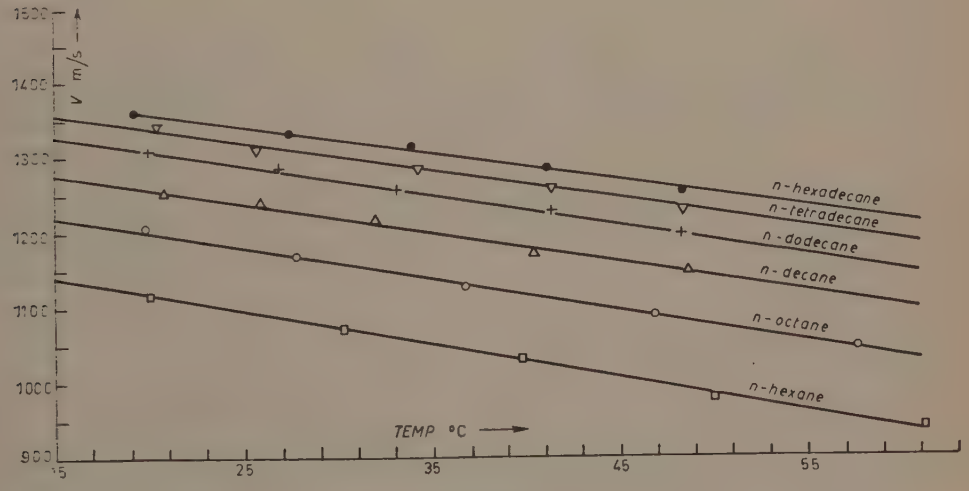

Fig. 10.

Fig. 12 shows the temperature variation of absorption in these liquids. It is seen that the temperature coefficient is positive for hexane and less positive for octane. For decane, the absorption is almost constant with tempe-

([24]) S. PARTHASARATHY and N. N. BAKHSHI: *Journ. Sci. Industr. Res.*, A-**12**, 448 (1953).

([25]) M. R. RAO: *Journ. Chem. Phys.*, **9**, 682 (1941).

([26]) W. SCHAAFFS: *Ergeb. Exakt. Naturwiss.*, **25**, 109 (1951).

rature in the range of measurements. The temperature coefficient then becomes negative and increases progressively from dodecane to hexadecane.

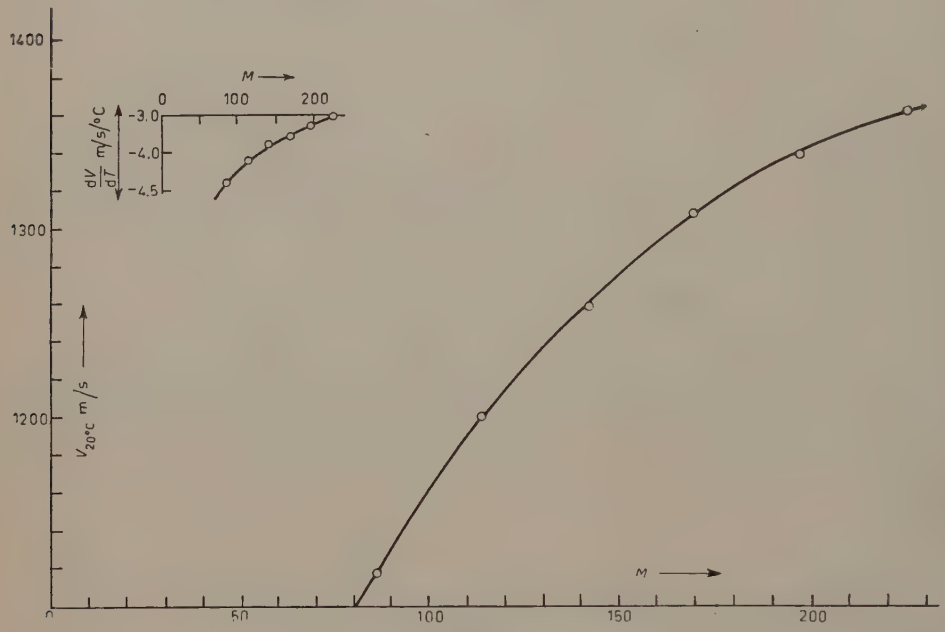

Fig. 11.

Fig. 13 shows the absorption at 20 °C as a function of molecular weight. The viscous absorption is also shown. It is seen that the absorption decreases at

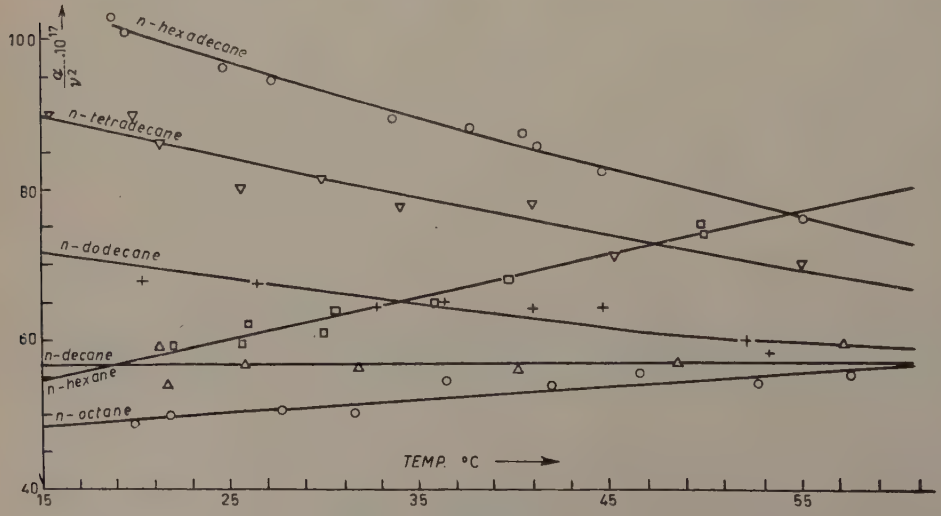

Fig. 12.

first, reaches a minimum value for octane and then increases. The observed absorption however always remains higher than the viscous absorption and

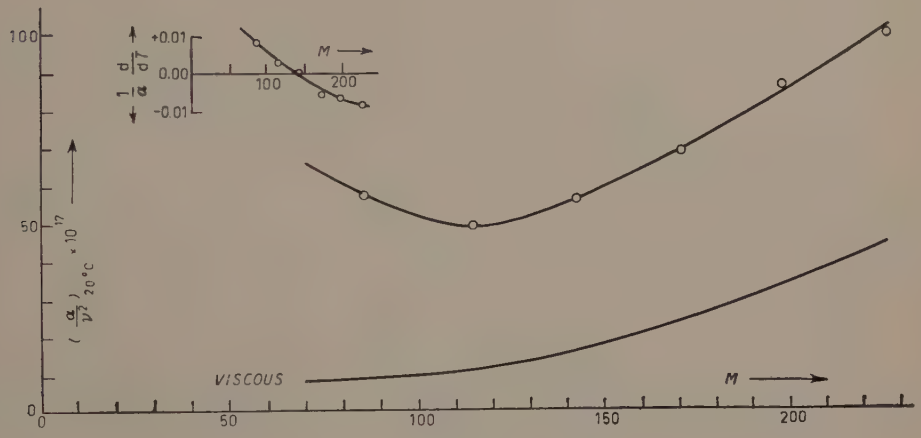

Fig. 13.

the two curves do not tend to come closer even at higher molecular weight. This behaviour is different from that in fatty acids where at higher molecular weight the observed absorption approaches the value of viscous absorption.

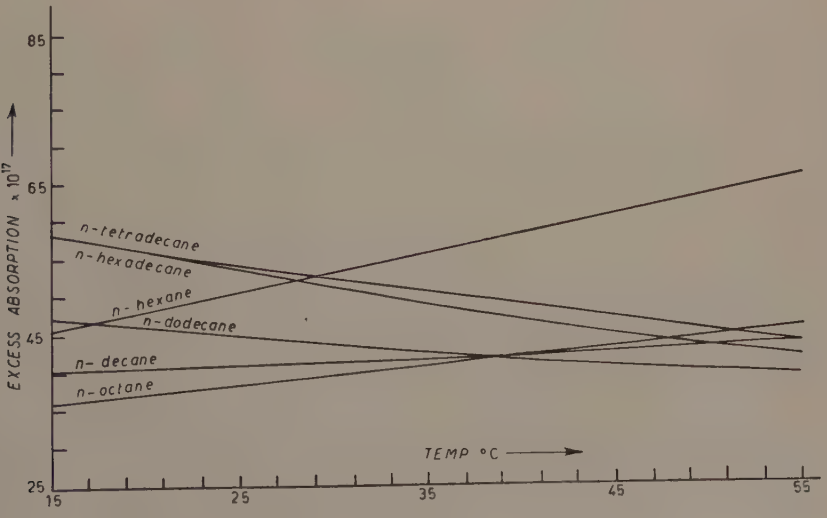

Fig. 14.

The difference between the two absorptions at different temperatures is shown in Fig. 14 and follows more or less the same trend as the observed absorption.

The ratio of the two absorption coefficients, shown in Fig. 15, is however different, increasing with temperature for the lowest member but becoming almost temperature independent for the higher members.

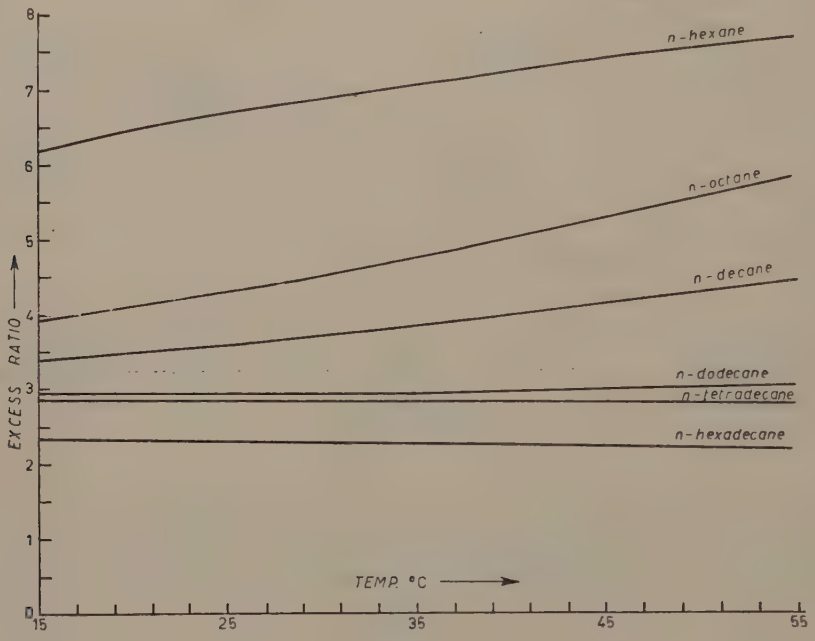

Fig. 15.

The values of absorption obtained here differ somewhat from those obtained earlier by PELLAM and GALT ([27]) and by KISHIMOTO and NOMOTO ([28]). They however agree with those of KOSHKIN and NOZDREV ([29]). The various results are shown in Fig. 16 and 17, along with the data for n-pentane by YOUNG and PETRAUSKAS ([30]). The latter workers have worked from — 100 °C to — 15 °C, while KOŠKIN and NOZDREV have worked from — 50 °C to 60 °C. These results show that in all these liquids the absorption is very high at low temperatures and falls rapidly with increasing temperature, reaching a minimum value, after which it rises again a little. It would thus appear that the temperature independent value of decane happens to be at this minimum region of absorption in the range of temperatures used in the present work. It is also possible that the higher members of the series would show a similar mi-

([27]) J. R. PELLAM and J. K. GALT: *Journ. Chem. Phys.*, **14**, 608 (1946).
([28]) T. KISHIMOTO and O. NOMOTO: *Journ. Phys. Soc. Japan*, **9**, 620 (1954).
([29]) N. I. KOŠKIN and V. F. NOZDREV: *Dokl. Akad. Nauk SSSR*, **92**, 793 (1953).
([30]) J. M. YOUNG and A. PETRAUSKAS: *Journ. Chem. Phys.*, **25**, 943 (1956).

nimum and reversal of temperature coefficient of absorption at a higher temperature.

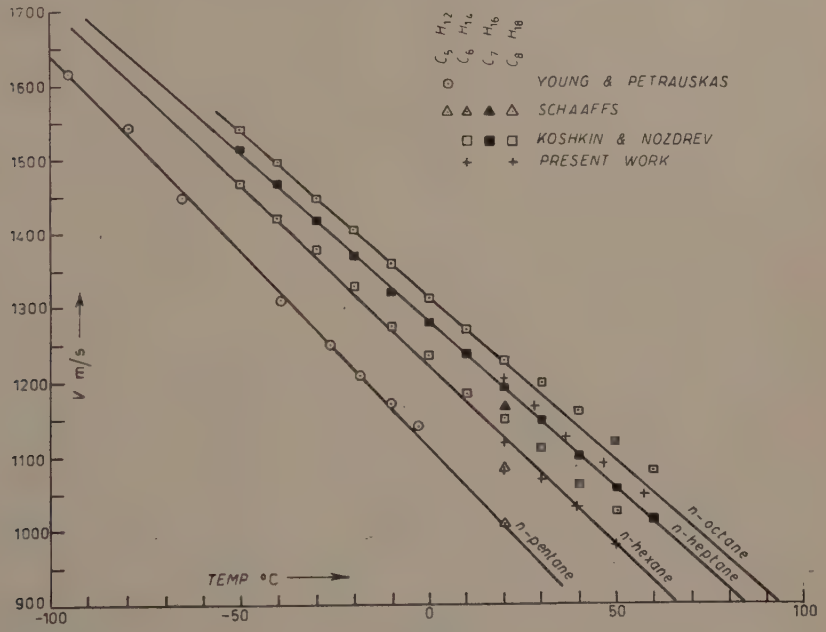

Fig. 16.

The minimum value of absorption is almost the same for all liquids *viz.* $55 \cdot 10^{-17}$ but the temperature at which it occurs differs from liquid to liquid.

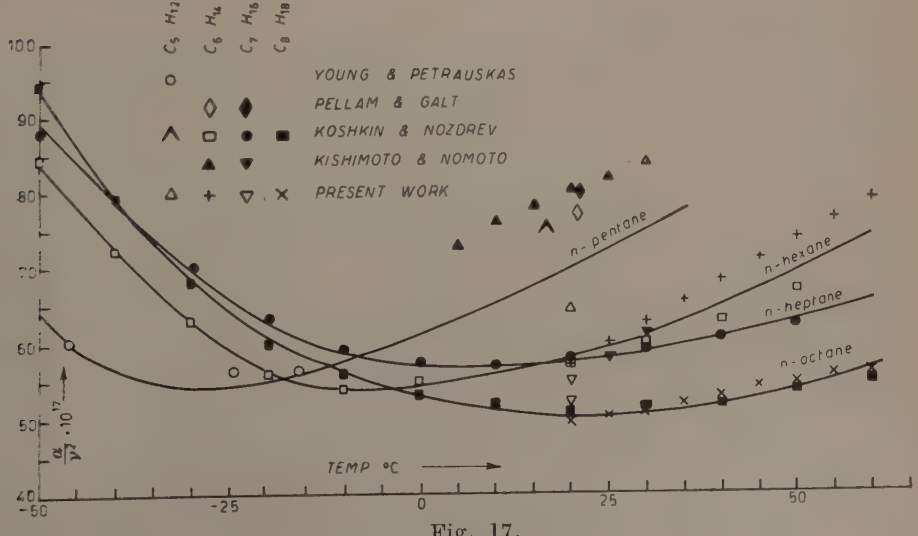

Fig. 17.

This temperature also shows a relationship to molecular weight as shown in Fig. 18. The viscous absorption calculated on this extended temperature region also behaves in the same way as the observed absorption, showing a minimum

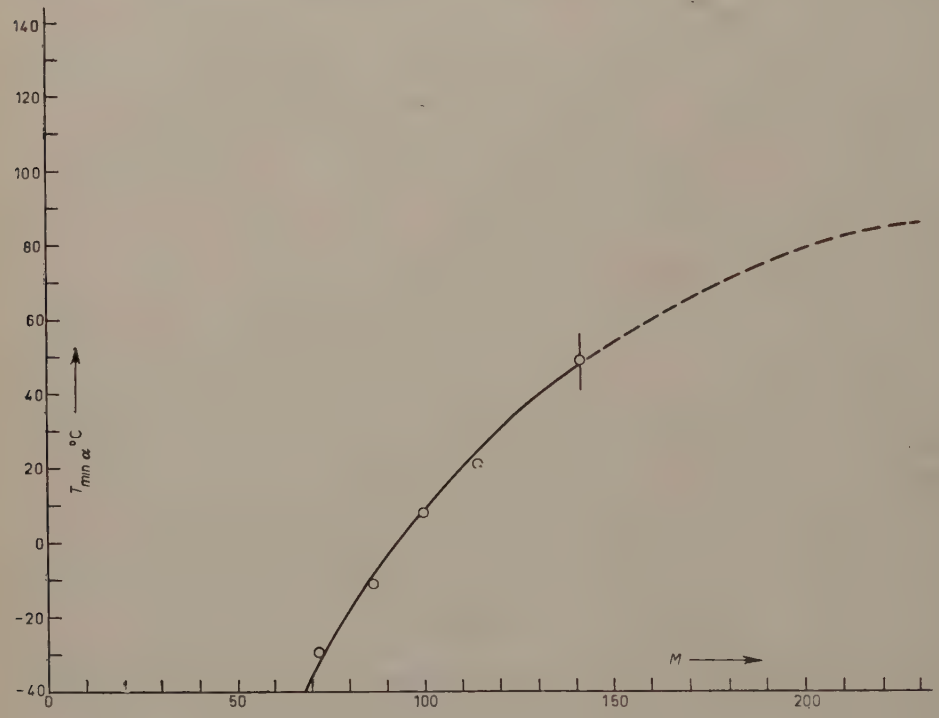

Fig. 18.

value at more or less the same temperatures. The excess absorption and ratio are also very high at low temperatures but decrease very rapidly with increasing temperature, the excess absorption reversing after reaching a minimum value and the ratio tending to a constant value.

The temperature coefficient of observed absorption in these liquids is about the same as that for the viscous absorption given by Stokes' theory, as seen in Table IV.

All these facts suggest that the cause of absorption is due to a viscosity effect. Since the observed absorption is always higher than the calculated viscous absorption, the excess can be ascribed either to bulk viscosity or to a relaxation in the quasi-crystalline structure, as suggested by HALL ([31]) in the case of water. The excess absorption is of the same order as that due to

([31]) L. HALL: *Phys. Rev.*, **71**, 318 (1947).

TABLE IV.

Liquid	Temperature coefficient of absorption	
	Observed	Calculated
n-Hexane	+ 0.014	+ 0.008
n-Octane	+ 0.003	+ 0.003
n-Dodecane	— 0.005	— 0.004
n-Tetradecane . . .	— 0.006	— 0.007
n-Hexadecane . . .	— 0.007	— 0.008

shear viscosity and its temperature dependence is also the same as that of shear viscosity, particularly for the higher members. For hexane, the two temperature coefficients are slightly different, and the excess absorption tends to increase with temperature. The fact that the temperature coefficient of absorption for hexane is strongly positive, as compared to the other members of the series where it is negative and of the same order as the coefficient for viscous absorption, suggests that there is an additional cause of absorption which may be thermal relaxation. In the higher members, the viscous effects predominate to give an overall absorption behaving like viscous absorption while for the lower members the other factor predominates and results in an opposite trend. For decane, these two causes balance out to give a temperature independent absorption.

At very low temperatures, the temperature coefficient for hexane is strongly negative and the excess absorption is very large. This is due to the fact that

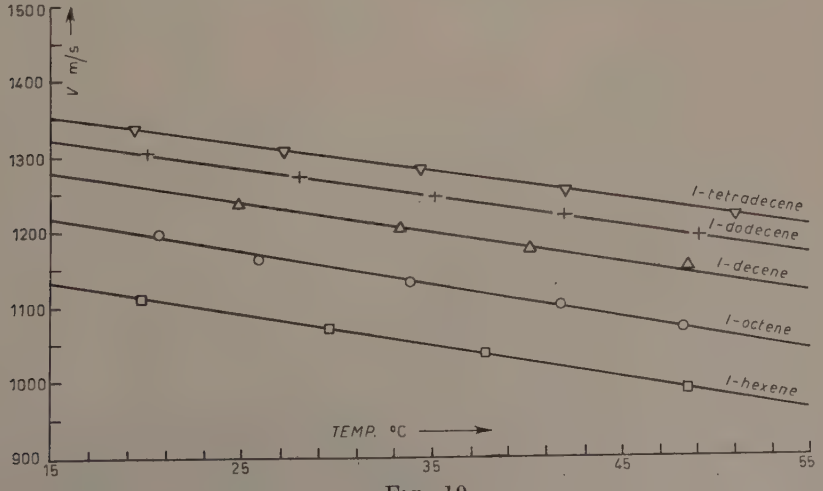

Fig. 19.

at these lower temperatures the effects of a possible quasi-crystalline structure are more predominant than at the higher temperatures where this kind of structure tends to break down.

It is therefore likely that in the liquids forming the aliphatic hydrocarbon series, the absorption is due to a combination of two effects, namely thermal relaxation and bulk viscosity or structural relaxation and the relative magnitudes of these depend upon the temperature, thus giving a positive, neutral or negative variation with temperature.

TABLE V. – *l-Olefines.*

Liquid	Velocity at 20 °C m/s	Temp. Coeff. of vel.	Observed absorption $\cdot 10^{17}$				$\frac{1}{\alpha}\frac{d\alpha}{dT}$
			20°	30°	40°	50 °C	
l-Hexene	1112	— 4.3	60.5	62.2	63.9	65.6	+ 0.0027
l-Octene	1194	— 4.2	57.6	57.6	57.6	57.6	0.0000
l-Decene	1259	— 3.7	51.0	48.2	45.6	43.1	— 0.0056
l-Dodecene	1302	— 3.6	71.2	65.7	60.6	56.0	— 0.0080
l-Tetradecene	1335	— 3.69	0.7	84.5	78.7	73.3	— 0.0071

Table V gives the velocity and absorption at different temperatures in liquids of this series. Fig. 19 and 20 show velocity as a function of tempe-

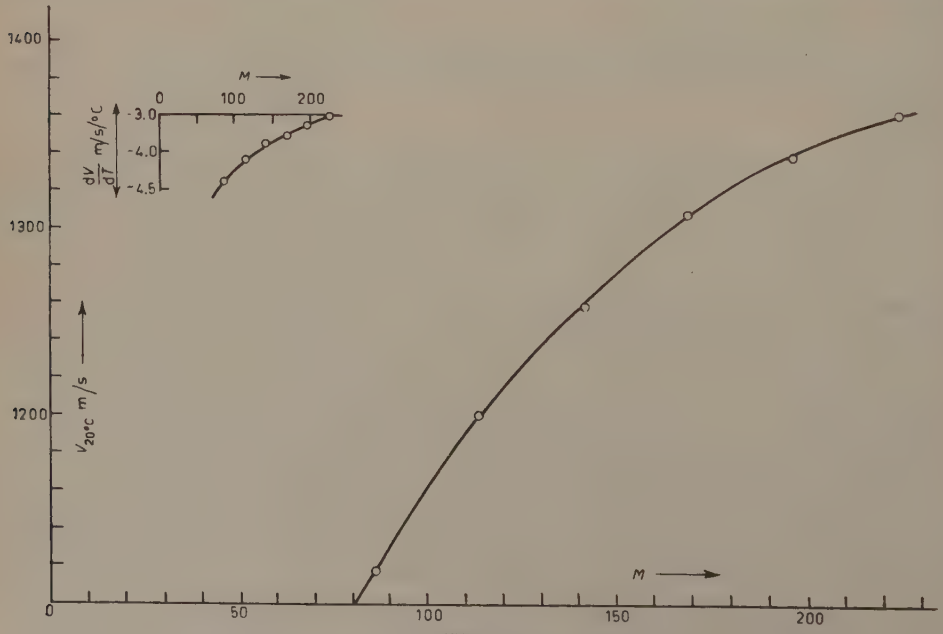

Fig. 2 .

rature and of molecular weight respectively. Fig. 21 shows the absorption at different temperatures, while Fig. 22 shows its relation to molecular weight.

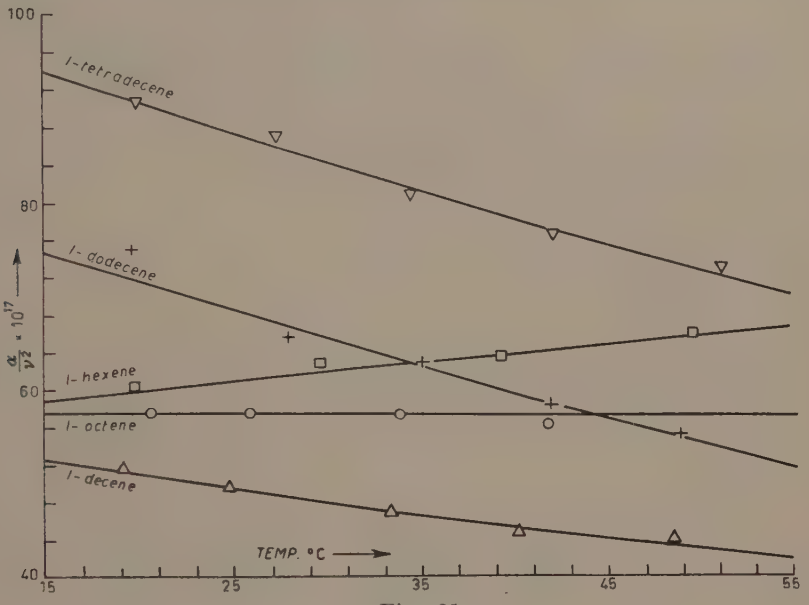

Fig. 21.

It is seen from these figures that the velocity and absorption, and their temperature coefficients, are similar to, and almost identical with, those for the

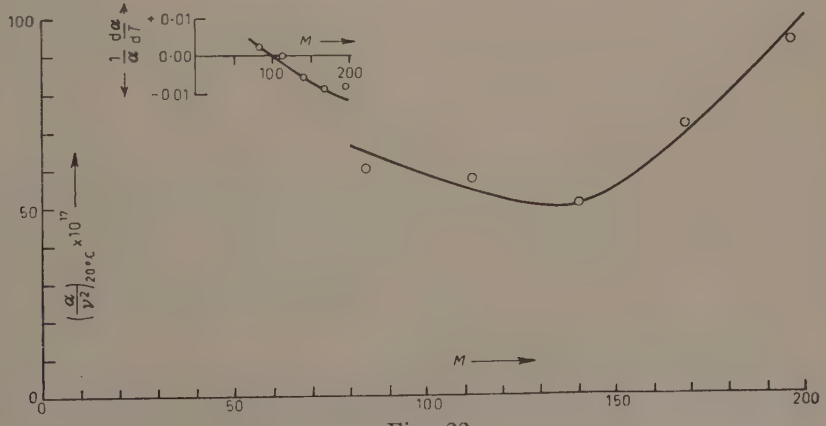

Fig. 22.

saturated hydrocarbons. Thus, here also, the absorption shows a decrease at first, reaching a minimum value for octane and then increasing for the

higher members. The temperature coefficient of absorption also starts with a positive value for hexane and turns negative for the higher members, with octane showing no temperature variation. Unfortunately the relevant physical data are not available for any liquids in this series and it is therefore not possible to calculate its viscous absorption. From the similarity between these two series, however, it may be expected that the trend in both the calculated absorption and excess absorption should be the same and that the causes of absorption are also of similar origin.

Aromatic hydrocarbons.

In order to compare the difference between the open-chain series of liquid hydrocarbons with those of the closed-chain, this series was used.

TABLE VI.

Liquid	Velocity at 20 °C m/s	Temp. Coeff. of vel.	Viscous absorption at 20 °C $\cdot 10^{17}$	Observed absorption $\cdot 10^{17}$				$\frac{1}{\alpha}\frac{d\alpha}{dT}$
				20°	30°	40°	50 °C	
Benzene	1323	— 5.2	8.4	850	910	956	1012	+0.0065
Toluene	1332	— 5.1	7.5	81	89	96	101	+0.0072
o-Xylene	1372	— 4.2	9.4	61	68	78	87	+0.0133
m-Xylene	1360	— 3.9	7.5	74	80	87	93	+0.0081
p-Xylene	1343	— 4.3	8.1	63	70	74	79	+0.0070

Benzene and toluene have been very thoroughly studied by several workers ([28,32-7]), hence the mean values are quoted here. For the three xylenes, no previous values are known and hence they have been measured here. Fig. 23 and 24 show the variation of velocity with temperature and with molecular weight respectively. Fig. 25 shows the variation of absorption with tempe-

([32]) W. SCHAAFFS: *Zeits. Phys. Chem.*, **194**, 28 (1944).
([33]) J. QUINN: *Journ. Acoust. Soc. Amer.*, **18**, 185 (1946).
([34]) D. SETTE: *Coll. over Ultrasonore Trillingen* (Brussel, 1951), p. 153.
([35]) G. S. VERMA: *Journ. Chem. Phys.*, **18**, 1352 (1950).
([36]) P. BAZULIN: *Compt. Rend. Acad. Sci. URSS*, **14**, 273 (1937).
([37]) H. GROBE: *Phys. Zeits.*, **39**, 333 (1938).

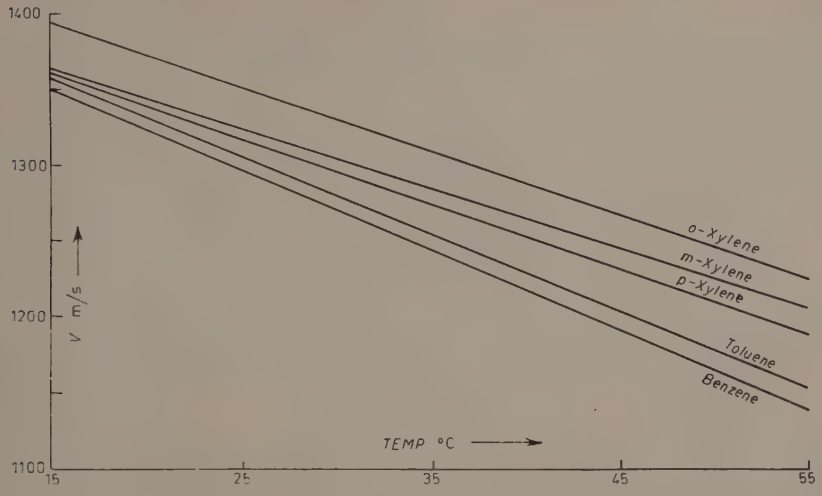

Fig. 23.

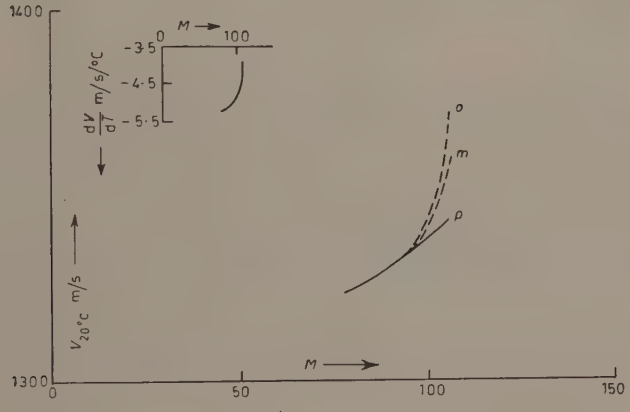

Fig. 24.

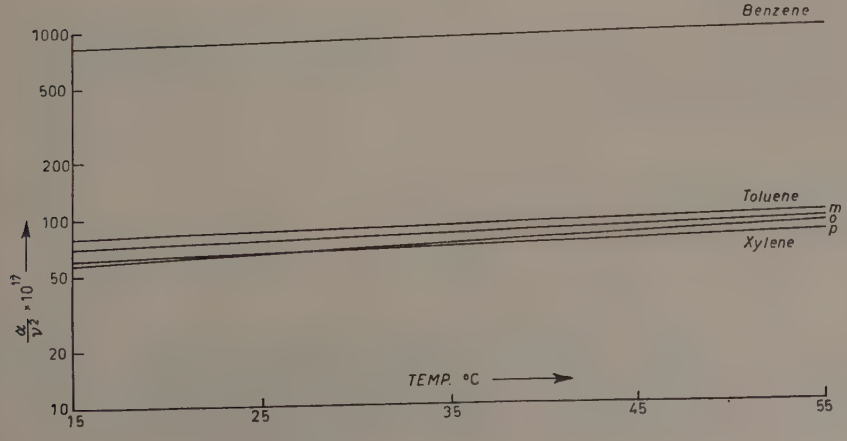

Fig. 25.

rature. Fig. 26 shows the relation of absorption and its temperature coefficient to the molecular weight. In this series, the temperature coefficients of absor-

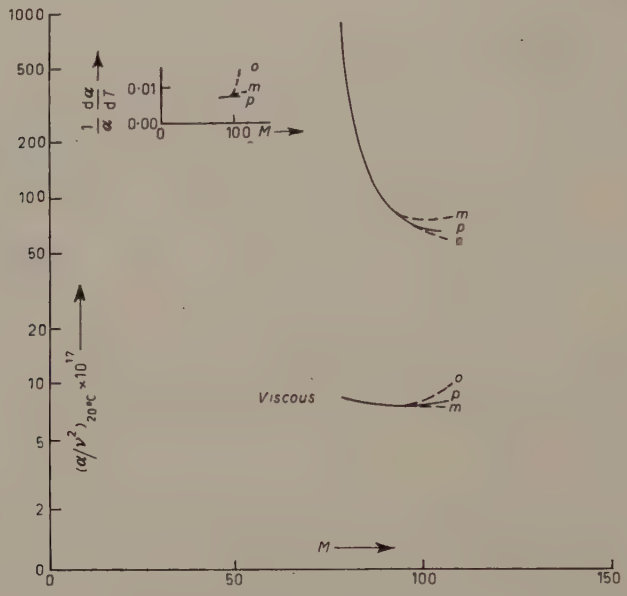

Fig. 26.

ption are all positive and the absorption of the first member is enormously higher than that of the other members, the latter having values comparable

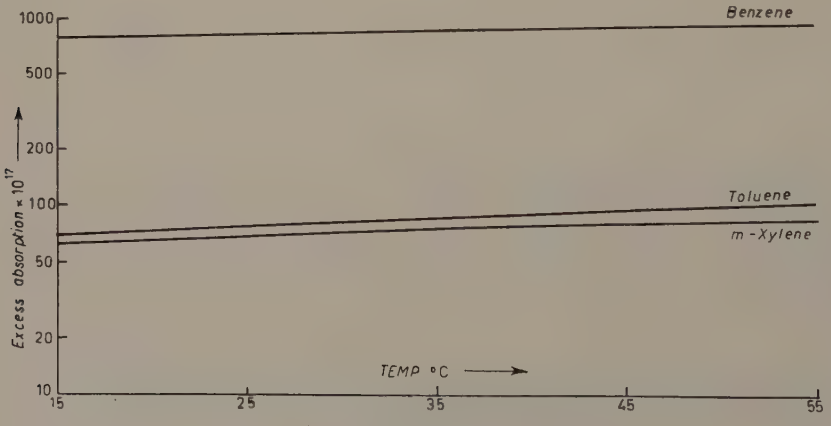

Fig. 27.

to each other. The excess absorption and excess ratio behave in the same way as the observed absorption, as shown in Fig. 27 and 28.

These results show that the cause of absorption in the liquids of this series is different from that in the other two series and probably results from thermal

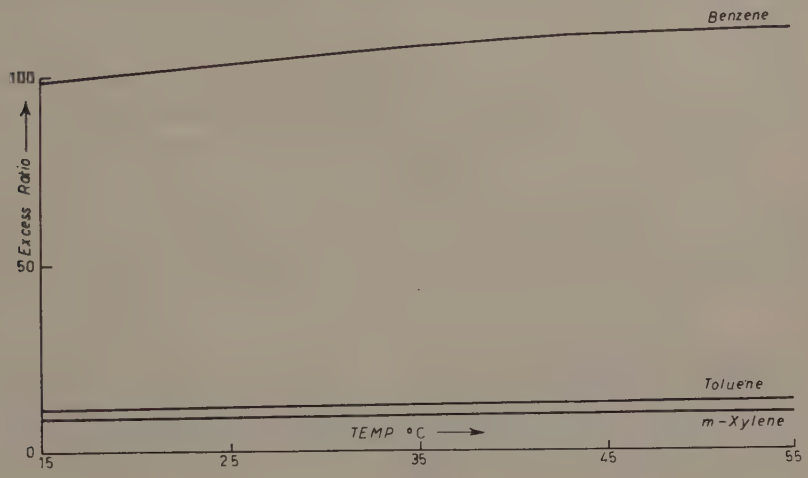

Fig. 28.

relaxation, the relaxation frequency being very much higher than the frequencies used for measurements.

RIASSUNTO

Gli Autori si sono proposti di dare un contributo alla interpretazione delle cause di assorbimento degli ultrasuoni nei liquidi, esaminando in particolare il problema dell'importanza relativa delle diverse cause di assorbimento (rilassamento termico, rilassamento strutturale, viscosità di volume; ecc.) in rapporto alla natura del liquido. A questo scopo essi hanno effettuato le loro determinazioni sperimentali su alcune serie omologhe di composti organici.

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IL NUOVO CIMENTO Vol. X, N. 1 1° Ottobre 1958

On the Universal Fermi Interaction.

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(ricevuto il 1° Luglio 1958)

Summary. — It is pointed out in connection with a universal Fermi interaction that conservation of axial vector current does not ensure non-renormalization of the axial vector coupling constant for strongly interacting particles. A discussion is given of possible renormalizing processes and their effects.

Current theoretical feeling [1,2] is in favour of a universal Fermi interaction between four Dirac particle fields A, B, C, D of the form

$$(\bar{A}\gamma_\mu(1-\gamma_5)B)(\bar{C}\gamma_\mu(1-\gamma_5)D)\,. \tag{1}$$

This universal A-V theory has had remarkable successes [2]. However, it has the difficulty associated with it that renormalization effects are expected to occur when the Fermions can interact strongly with some other field. It was because of this difficulty and in order to account for the excellent agreement between the coupling constants for μ-meson and ^{14}O decay, that Feynman and Gell-Mann [1] proposed that the V-part of the interaction be described in terms of a conserved 4-current J^V_μ. That is, they required that

$$\partial J^V_\mu/\partial x_\mu = 0\,. \tag{2}$$

Now since the main contribution to the V-part of the decay of a neutron comes from J^V_4 which transforms like a density, it is clear by comparison with

(*) Part of this work was performed whilst the author was vacation consultant at A.E.R.E., Harwell, Didcot, Berkshire.

(1) R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958).

(2) R. E. Marshak and E. C. G. Sudarshan: to be published.

electrodynamics that (2) ensures that pion renormalization effects, for example, should disappear. For, in electrodynamics in the low energy limit (which is the situation we have here), there is no renormalization of the coupling constant e due to mesonic effects. That is, the total charge is not changed. The only difference between the two cases is that in electrodynamics we are interested in the charge and therefore in the z-component of the isotopic spin operator whereas in β-decay it is the $\pm$-components that are relevant. Further, just as there are no mesonic exchange contributions to the total charge of a nucleus in electrodynamics so there should be no exchange contributions in the β-decay of ^{14}O for example ([3]). This means that use of the $^{14}O \rightarrow {}^{14}N$ decay to determine the strength of the V-interaction is perfectly correct within the framework of this theory.

FEYNMAN and GELL-MANN ([1]) also hazard the opinion that it may be that there is no renormalization of the A-part of the universal Fermi interaction. Recently, with this point in mind, POLKINGHORNE ([4]) has shown that, by introducing Schwinger's σ-particle ([5]), it is possible to construct an axial vector current J_μ^A satisfying

$$\partial J_\mu^A / \partial x_\mu = 0 \,. \tag{3}$$

It is the primary object of this note to point out that even if J_μ^A satisfies the continuity equation (3), this does not ensure that there will be no renormalization of the axial vector coupling constant for nucleons due to pion interactions (*). The essential point is that the main contribution to the A-part of the decay of a neutron comes from the three *spatial* components of J_μ^A rather than the *time* component so that, although for the polar vector case the continuity equation guarantees the conservation of « total charge », in the axial vector case it does not guarantee the conservation of « total spatial current ». Conversely, consider the case of a nucleon but with the strong interactions « switched off ». Under these circumstances there will clearly be no renormalization effects and yet for such a particle the field equations lead to

$$\frac{\partial}{\partial x_\mu} (\bar{N} \gamma_5 \gamma_\mu \tau_i N) = - 2 m^0 \bar{N} \gamma_5 \tau_i N \,, \tag{4}$$

implying that the axial vector current is *not* conserved. Here m^0 is the bare mass of a nucleon).

([3]) J. S. BELL and R. J. BLIN-STOYLE: *Nuclear Physics*, **6**, 87 (1958).
([4]) J. C. POLKINGHORNE: *Nuovo Cimento*, **8**, 179 (1958).
([5]) J. SCHWINGER: *Ann. of Phys.*, **2**, 407 (1957).
(*) It has come to the author's notice that Dr. T. H. R. SKYRME has independently come to the same conclusion.

Nevertheless we can still enquire whether it is possible in some other general fashion to arrange that there shall be no renormalization of the axial vector coupling constant. An indication that this is not so is obtained by considering the situation in a particularization of the general case, namely the lowest order non-relativistic perturbation approximation. In this approximation it is a straightforward matter to calculate the expression for renormalization of the axial vector coupling constant due to pion emission as an integral in momentum space ([6]) and to compare it with the corresponding expressions calculated for other possible renormalizing processes (*e.g.* introducing a σ-particle as suggested by POLKINGHORNE; taking account of strange particle interactions). In no case considered does the momentum integral have the same form as that associated with pion renormalization. For example if the interaction of σ and π with nucleons is

$$\sqrt{4\pi}\, g(\bar{N}\gamma_5 \boldsymbol{\tau} N \cdot \boldsymbol{\pi} + \bar{N} N \sigma) \tag{5}$$

and J_μ^A is taken to have the form

$$J_\mu^A = \left[i\bar{N}\gamma_5\gamma_\mu\tau_i N + C\left(\pi_i \frac{\partial\sigma}{\partial x_\mu} - \frac{\partial\pi_i}{\partial x_\mu}\sigma\right)\right], \tag{6}$$

where C is an arbitrary constant, then the renormalized axial vector coupling constant g_A' is related to the unrenormalized constant g_A by

$$g_A' = g_A[1 + \Delta_\pi + \Delta_\sigma] \tag{7}$$

where

$$\Delta_\pi = -\frac{2}{3\pi} g^2 \int_0^{k_{\max}/M} \frac{x^4 \mathrm{d}x}{(\alpha^2 + x^2)^{\frac{1}{2}}}, \qquad \Delta_\sigma = +\frac{2g^2}{3\pi} C \int_0^{k_{\max}/M} \frac{x^4 \mathrm{d}x}{(\alpha^2 + x^2)(\beta^2 + x^2)},$$

and $\alpha = m_\pi/M$, $\beta = m_\sigma/M$, M = nucleon mass and a momentum cut-off at $k_{\max}$ has been introduced. Exact cancellation between Δ_π and Δ_σ for arbitrary $k_{\max}$ can clearly not take place, and in general it does not seem possible to ensure exact non-renormalization of the A-part of the interaction.

On the other hand, if we assume that there is no renormalization of the V-part, then from the known limits on the *ft*-value for the decay of a neutron it is possible to obtain an estimate for the renormalization of the A-part. Taking the universal interaction for the bare neutron to be V-A and the

([6]) R. J. FINKELSTEIN and S. A. MOSZKOWSKI: *Phys. Rev.*, **95**, 1695 (1954); S. S. GERSCHTEIN and IA. B. ZEL'DOVICH: *Soviet Physics* (*JETP*), **2**, 576 (1956).

strength of the V-part from the ^{14}O data it then follows that for the physical neutron the interaction is $V - \eta A$ where $1{\cdot}21 < \eta < 1{\cdot}28$. Thus experiment bears out the above suggestion that there is some renormalization of A-part.

Insofar as the renormalization takes the form of an enhancement of the axial vector interaction, the implications are that there are large contributions from other processes in addition to the purely pion interactions since all estimates (7) taking account only of pion interactions, give a reduction with $\eta \approx 0.38$ to 0.55. Typical processes might be the results of:

a) The nucleon-nucleon-σ interaction linked with a term in J_μ^A of the form $(\pi_i(\partial\sigma/\partial x_\mu) - (\partial\pi_i/\partial x_\mu)\sigma)$. This is the example previously discussed and it is interesting to note that if the momentum integrals in (7) are cut off at $k_{max} = M$ (*i.e.* $x = 1$), then for $C = 1$, for example, $g_A' > g_A$ as required. Of course the perturbation theory is too crude to warrant any quantitative estimate of the renormalization being made but the qualitative result probably has some significance.

b) The nucleon-Σ-K interaction linked with a term in J_A^μ of the form $(i\bar{\Sigma}\gamma_5\gamma_\mu T_i \Sigma)$. A no-recoil perturbation approximation calculation for this case indicates that, taking the nucleon and Σ to have same parity, a scalar K leads to enhancement and a pseudo-scalar to reduction.

However it seems unlikely that *b*) or any combination of processes involving strange particles can account for the overall enhancement since the K-interactions are probably a factor 10 weaker than the pion interactions (8). This being so, it seems that the σ-particle must be invoked as in *a*) or the whole concept of a universal V-A interaction abandoned.

* * *

The author is indebted to Dr. J. S. Bell for comments on this work.

(7) M. Ross: *Phys. Rev.*, **104**, 1736 (1956).
(8) M. Gell-Mann: *Proceedings of the Rochester Conference* (1957), IX-6.

RIASSUNTO (*)

Si fa rilevare a proposito della interazione universale di Fermi che la conservazione della corrente di vettore assiale non assicura la non rinormalizzazione della costante d'accoppiamento del vettore assiale per particelle fortemente interagenti. Segue una discussione sui possibili processi di rinormalizzazione e dei loro effetti.

(*) *Traduzione a cura della Redazione.*

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Capture of μ-Mesons in Deuterium.

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CERN - Geneva

(ricevuto il 1° Luglio 1958)

Summary. — The neutron spectrum, the neutron asymmetry with respect to muon spin direction, and the neutron polarization are calculated as functions of neutron energy for μ-meson capture in deuterium. The total capture rate and the dependence of the rate on the hyperfine state of the mesic atom are given and compared to approximate expressions. The dependence of the observations on the amount of Fermi or Gamow-Teller coupling and on the presence of an effective pseudoscalar coupling is displayed.

1. – Introduction.

The least well-known of the weak interactions is that responsible for the capture of μ-mesons by nuclei. Recently a number of papers ([1,2]) have discussed the possible observations on the capture in hydrogen: 1) the capture probability, 2) the neutron asymmetry with respect to the muon spin direction, and 3) the neutron polarization. We have calculated these observables as functions of the energy of one neutron for the case of capture in deuterium. The difference in the total capture rates from the different hyperfine levels ([3]) was also obtained. Capture in dueterium is of particular interest ([4]), because

(*) On leave of absence from Carnegie Institute of Technology, Pittsburgh, Pa. USA.

([1]) I. S. Shapiro, E. I. Dolinsky and L. D. Blohincev: *Nucl. Phys.*, **4**, 273 (1957); K. Huang, C. N. Yang and T. D. Lee: *Phys. Rev.*, **108**, 1340 (1957).

([2]) L. Wolfenstein: *Nuovo Cimento*, **7**, 706 (1958).

([3]) J. Bernstein, T. D. Lee, C. N. Yang and H. Primakoff: *Phys. Rev.*, **111**, 313 (1958).

([4]) A. Rudik: *Dokl. Akad. Nauk SSSR*, **92**, 739 (1953); H. Primakoff: *Proc. of the Fifth Annual Rochester Conference on High Energy Physics* (New York, 1955), p. 174 and private communication.

the effect of the exclusion principle in the final state, which consists of two neutrons, is to depress the capture rate due to the Fermi coupling and thus to enhance the relative importance of Gamow-Teller couplings. Comparison between capture in hydrogen and deuterium thus may serve to distinguish between the two types of coupling. In particular, it may be noted that for equal values of Fermi and Gamow-Teller couplings, the neutron asymmetry vanishes in the non-relativistic approximation for capture in hydrogen, but would not for capture in deuterium. While these observations are very difficult to perform in deuterium and some are probably impossible, the calculation of capture in deuterium may also serve to illustrate some qualitative effects characteristic of capture in heavier nuclei.

The calculation is based on the following assumptions and approximations: 1) the μ-meson capture by a proton is described by a local four-field interaction as in the theory of β-decay, 2) the μ-meson and the nucleons are treated non-relativistically, 3) the ground state of the deuteron is described by a HULTHÉN function with the D-state ignored, and 4) the neutrons in the final state interact only in S-states and this interaction may be described in the effective-range approximation. The effect of these approximations is discussed in Sect. 5. In the non-relativistic limit there are two coupling constants: the Fermi G_F and the Gamow-Teller G_G; we include also (for the neutron spectrum and asymmetry) the pseudoscalar coupling C_P because it has been shown [5] that virtual pion effects lead to a large effective coupling of this type.

2. – Formulae.

The matrix element for the transition may be written using perturbation theory ($\hbar = c = 1$ throughout)

$$(1) \qquad H_i = \int \exp\left[-i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_2)/2\right] F_f^*(\boldsymbol{q},\boldsymbol{r}) J_f^\dagger \chi_f^\dagger [\tau_-(1)V(1) + \tau_-(2)V(2)] J_d \chi_d \psi_d(r)\, \mathrm{d}\boldsymbol{r}_1\, \mathrm{d}\boldsymbol{r}_2 ,$$

where

$\boldsymbol{K} = \boldsymbol{p}_1 + \boldsymbol{p}_2$	momentum of centre-of-mass of final two neutrons,
$F_f(\boldsymbol{q},\boldsymbol{r})$	wave function of relative motion of final two neutrons,
$\psi_d(r)$	deuteron space wave function,
χ_d, χ_f	initial, final spin functions of nucleons,
J_d, J_f	initial, final i-spin functions of nucleons,
$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$,	
$\boldsymbol{q} = \frac{1}{2}(\boldsymbol{p}_1 - \boldsymbol{p}_2)$	relative momentum of final two neutrons,

[5] L. WOLFENSTEIN: *Nuovo Cimento*, **8**, 882 (1958).

$$V(i) = \exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i](A + \boldsymbol{B}\cdot\boldsymbol{\sigma}_i)\varphi_\mu(\boldsymbol{r}_i)\,, \tag{2}$$

$\boldsymbol{\nu}$ neutrino momentum,

$\varphi_\mu(\boldsymbol{r})$ ground state wave function of μ-mesic atom of deuterium.

$$\sqrt{2}\,A = G_F^*\psi_\nu^\dagger\psi_\mu + G_F'^*\psi_\nu^\dagger\gamma_5\psi_\mu\,,$$

$$\sqrt{2}\boldsymbol{B} = G_G^*\psi_\nu^\dagger\boldsymbol{\sigma}\psi_\mu + G_G'^*\psi_\nu^\dagger\gamma_5\boldsymbol{\sigma}\psi_\mu - \frac{C_P^*}{2M}\psi_\nu^\dagger\boldsymbol{\nu}\gamma_5\psi_\mu - \frac{C_P'^*}{2M}\psi_\nu^\dagger\boldsymbol{\nu}\psi_\mu\,,$$

ψ_ν, ψ_μ neutrino, muon spinor,

$$G_F = C_S + C_V\,, \qquad G_F' = C_S' + C_V'\,,$$

$$G_G = C_A + C_T\,, \qquad G_G' = C_A' + C_T'\,.$$

We use the hermitean conjugate of the interaction given by LEE and YANG ([6]), with e replaced by μ; the form of A and $\boldsymbol{B}$ depends upon the non-relativistic approximation for ψ_μ.

Substituting (2) in (1) and neglecting the variation of φ_μ over the deuteron, we get

$$\begin{cases} H_i = (2\pi)^3\delta(\boldsymbol{K}+\boldsymbol{\nu})\varphi_\mu(0)M_f\,, \\ M_f = \sqrt{2}\chi_f(A + \boldsymbol{B}\cdot\boldsymbol{\sigma}_1)\chi_a\int F_f^*(\boldsymbol{q},\boldsymbol{r})\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}/2]\psi_a(r)\,\mathrm{d}\boldsymbol{r}\,. \end{cases} \tag{3}$$

The transition probability for the case that one neutron is observed is given by

$$\begin{cases} \mathrm{d}w_\lambda = \mathrm{d}\boldsymbol{p}_1(2\pi)^{-6}(\pi a_\mathrm{D}^3)^{-1}\int 2\pi\sum_\nu |M_f|^2\delta(\boldsymbol{K}+\boldsymbol{\nu})\,. \\ \delta\left(\mu - D - \nu - \frac{\nu^2}{4M} - \frac{q^2}{M}\right)\mathrm{d}\nu\,\mathrm{d}\boldsymbol{p}_2\,, \end{cases} \tag{4}$$

where the sum is over neutrino spins;

μ mass of μ-meson;

D mass of two neutrons minus mass of deuteron;

a_D Bohr radius of μ-mesic atom of deuterium.

([6]) T. D. LEE and C. N. YANG: *Phys. Rev.*, **104**, 254 (1956), Eq. (A.1). The two-component neutrino theory is obtained by setting $G_l' = G_l$ for left-handed neutrinos and $G_l' = -G_l$ for right-handed ones. For parity-conserving processes, the two-component neutrino theory is identical with the old parity-conserving interaction provided Eq. (A.1) is divided by $\sqrt{2}$, which we have done.

The subscript λ stands for one of the six possible initial spin states (two for the μ-meson, three for the deuteron) and one of the four possible final spin states of the two nucleons. The results of interest are obtained by summing over the spin directions of the unobserved neutron plus any other spin directions not observed:

1) The probability of observing a neutron after the capture process with energy E is

$$\mathrm{d}w = \frac{1}{6}\sum_{\lambda}\mathrm{d}w_{\lambda} = \frac{1}{\pi a_{\mathrm{D}}^{3}}\,\frac{M^{2}\,\mathrm{d}E}{4(2\pi)^{4}}\,\mathrm{d}\Omega I_{0}\,, \tag{5}$$

$$I_{0} = a_{FF}I_{tt} + (a_{GG} - \tfrac{2}{3}\,\mathrm{Re}\,a_{GP})(2I_{tt} + I_{ss})\,. \tag{5a}$$

2) The angular distribution of the neutrons is of the form $(1 - PA\cos\theta)$ where θ is the angle between the neutron momentum $\boldsymbol{p}$ and the μ-meson spin, P the degree of polarization of the μ-meson at the moment of capture, and

$$I_{0}A = b_{FF}I'_{tt} - \tfrac{1}{3}(b_{GG} + 2\,\mathrm{Re}\,b_{GP})(2I'_{tt} + I'_{ss})\,. \tag{5b}$$

3) The spin polarization $\langle\boldsymbol{\sigma}\rangle$ of the neutron (omitting the contribution of C_{P}) is given by

$$\left\{\begin{aligned} &I_{0}(1 - PA\cos\theta)\langle\boldsymbol{\sigma}\rangle = -a\boldsymbol{p}/p + Pc\boldsymbol{s} - Pd(\boldsymbol{p}\times\boldsymbol{s})/p\,,\\ &-a = \tfrac{2}{3}b_{GG}(I'_{tt} + 2\,\mathrm{Re}\,I'_{st}) - \tfrac{2}{3}\,\mathrm{Re}\,[b_{GF}(2I'_{tt} + I'_{st})]\,,\\ &\quad c = \tfrac{2}{3}a_{GG}(I_{tt} + 2\,\mathrm{Re}\,I_{st}) + \tfrac{2}{3}\,\mathrm{Re}\,[a_{GF}(2I_{tt} + I_{st})]\,,\\ &-d = \tfrac{2}{3}\,\mathrm{Im}\,(b_{GF}I'_{st}) + \tfrac{4}{3}\,\mathrm{Im}\,(b_{GF}I'_{tt})\,, \end{aligned}\right. \tag{5c}$$

where $\boldsymbol{s}$ is the unit vector in the direction of the μ-meson spin.

4) The difference in the capture rate between the hyperfine states $F = \frac{3}{2}$ and $F = \frac{1}{2}$ of the μ-mesic atom of deuterium is (omitting the C_{P} term again)

$$\frac{\lambda_{+} - \lambda_{-}}{3\bar{\lambda}} = \frac{2\,\mathrm{Re}\,a_{GF}\int I_{tt}\,\mathrm{d}E - a_{GG}\int(I_{tt}+I_{ss})\,\mathrm{d}E}{a_{FF}\int I_{tt}\,\mathrm{d}E + a_{GG}\int(2I_{tt} + I_{ss})\,\mathrm{d}E}\,, \tag{6}$$

where λ_{+}, λ_{-} are the capture probabilities per second from the quartet and the doublet hyperfine state, $\bar{\lambda}$ the average capture probability, $\bar{\lambda} = \frac{2}{3}\lambda_{+} + \frac{1}{3}\lambda_{-}$,

and

$$I_{ij} = \int_{\nu_{\min}}^{\nu_{\max}} J_i^* J_j \nu \, \mathrm{d}\nu \,,$$

$$I'_{ij} = \int_{\nu_{\min}}^{\nu_{\max}} J_i^* J_j \frac{\boldsymbol{\nu}\cdot\boldsymbol{p}}{\nu p} \nu \, \mathrm{d}\nu \,,$$

$$J_j = \int F_j^* \left(\boldsymbol{p} + \frac{\boldsymbol{\nu}}{2}, \boldsymbol{r}\right) \exp\left[-i\boldsymbol{\nu}\cdot\boldsymbol{r}/2\right] \psi_i(r) \, \mathrm{d}\boldsymbol{r} \,,$$

where j is either singlet (s) or triplet (t),

$$a_{lm} = G_l G_m^* + G'_l G'^*_m \,,$$

$$b_{lm} = G'_l G_m^* + G_l G'^*_m \,,$$

with $l,\ m = (F, G, P)$,

$$G_P = C_P \frac{\bar{\nu}}{2M} \,, \qquad G'_P = C'_P \frac{\nu}{2M} \,,$$

and from energy conservation,

$$(\boldsymbol{\nu}\cdot\boldsymbol{p}) = M(\mu - D - \nu) - \tfrac{1}{2}\nu^2 - p^2 \,.$$

In treating the pseudoscalar interaction, we have approximated the magnitude of the neutrino momentum by a constant $\bar{\nu}$. This is particularly justified when we consider the effective pseudoscalar interaction ([5]) due to a virtual pion since in this case the proportionality to ν is greatly cancelled by the $(q^2 + m_\pi^2)^{-1}$ dependence of C_P, where q is the four-momentum transfer. We have omitted terms in $|C_P|^2$ which should be of the same order as neglected relativistic terms.

We note that for no interaction (finally or initially) between the nucleons, Eqs. (5) reduce to those previously given ([2]) for capture in hydrogen (*). If the final-state interaction is ignored, the integrals can be expressed in closed form. The final state S-wave interaction has been included by numerical integration (+) in the following way.

(*) Eq. (1) of Ref. ([2]) should be multiplied by a factor $(1+\nu/M)^{-1}$.

(+) Tables of the integrals I_{ij} and I'_{ij} are available from the authors.

The singlet wave function of the final two-neutron system, $F_s(\boldsymbol{q}, \boldsymbol{r})$, was taken ([7]) as a symmetrized plane wave, plus the interacting S-state term

$$2[F_0(\boldsymbol{q}, \boldsymbol{r}) - (1/qr) \sin qr] .$$

The corresponding integrals can approximately be evaluated in terms of the effective ranges r_t of the deuteron and r_s of the two-neutron system, by separating F_0 from its asymptotic expression (plane wave plus incoming spherical wave),

$$F_0 = (1/qr) \sin (qr + \delta) \exp [-i\delta] + [F_0 - (1/qr) \sin (qr + \delta) \exp [-i\delta]]$$

and using a procedure of BETHE and LONGMIRE ([8]). The deuteron ground state is described by a Hulthén function

$$\psi_d(r) = (1/r)(\exp [-\alpha r] - \exp [-\beta r]) ,$$

and the effective range approximation

$$q \cot \delta = -(1/a) + \tfrac{1}{2} r_s q^2$$

is made. For the occurring parameters, we used the following numerical values:

$\alpha = (MD)^{\frac{1}{2}} = 45.7$ MeV

$\beta = 287$ MeV — (this is fitted to give the experimental deuteron effective range),

$a = -1.70 \cdot 10^{-12}$ cm — (taken as the equivalent proton-proton scattering length ([9]), which agrees within the limits of error with the neutron-neutron scattering length as determined by PHILLIPS and CROWE ([10]) from pion capture in deuterium),

$r_t = 1.70 \cdot 10^{-13}$ cm,

$r_s = 2.65 \cdot 10^{-13}$ cm — (assumed to be equal to the proton-proton singlet effective range).

([7]) K. M. WATSON and R. N. STUART: *Phys. Rev.*, **82**, 738 (1951).
([8]) H. A. BETHE and C. LONGMIRE: *Phys. Rev.*, **77**, 647 (1950).
([9]) J. M. BLATT and V. F. WEISSKOPF: *Theoretical Nuclear Physics* (New York, 1952).
([10]) R. H. PHILLIPS and K. M. CROWE: *Phys. Rev.*, **96**, 484 (1954).

3. – Discussion of results.

3·1. *Neutron spectrum.* – The neutron spectrum is shown in Fig. 1 for a pure Fermi and a pure Gamow-Teller (plus pseudoscalar, if desired) interaction: the latter is shown with and without the final ^1S state interaction (*). As discussed by RUDIK (4), the Fermi interaction yields a double peak in the spectrum: a broad peak around 5 MeV ($\sim \mu^2/2M$) corresponding to the neutrons resulting directly from capture, and a sharper peak below 0.5 MeV corresponding to the neutron left behind. These peaks are merged into a single peak around 1.3 MeV ($\sim \mu^2/8M$) in the Gamow-Teller case because of the predominant role of the ^1S state in which the particles have a strong attractive interaction. It is clear that the spectrum for an equal mixture of Fermi and Gamow-Teller interactions is practically indistinguishable from that of a pure Gamow-Teller interaction. On the other hand, a small admixture of Gamow-Teller would display itself by the presence of this peak. The pseudoscalar interaction, with the approximation employed, serves simply to enhance or weaken the contribution of the Gamow-Teller interaction without changing its spectrum. Because of the weakness of the capture interaction, μ-meson capture has an advantage over π-meson capture as a means of determining the neutron-neutron final state interaction (7,10). We have made calculations for an alternative value of

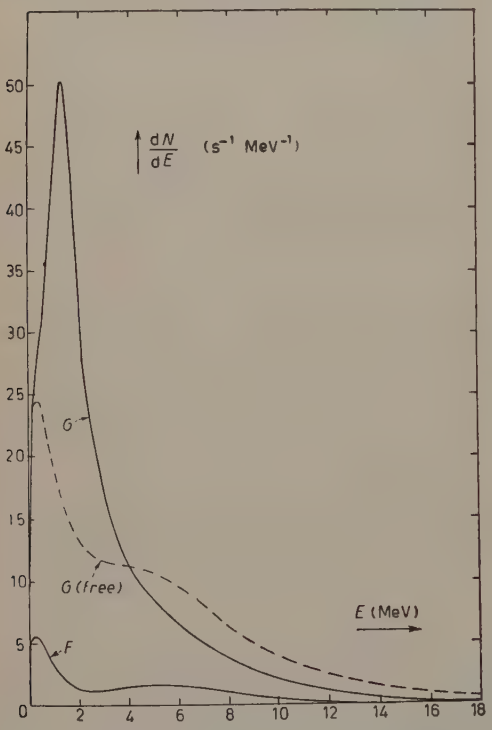

Fig. 1. – Neutron spectrum for pure Fermi and Gamow-Teller transitions, (the latter with and without final state interactions), normalized by using two-component neutrino theory and the Fermi coupling constant of β-decay, $G = 1.41 \cdot 10^{-49}$ erg cm³.

(*) The absolute rate is obtained by setting $a_{FF} = 2G^2$ and $a_{GG} - \frac{2}{3}a_{GP} = 2G^2$, where G is the experimental Fermi coupling constant (11) of β-decay, $G = 10^{-5}/M^2$. The curves omit the high-energy tail of the neutron spectrum which extends to 50 MeV ($= \frac{1}{2}(\mu - D)$).

(11) This value has been emphasized by R. P. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1958).

the ^1S scattering length, namely $a = -2.43 \cdot 10^{-12}$ cm, which corresponds to the experimental neutron-proton scattering length, and find it differs from the curve given only in causing the maximum at 1.3 MeV to be larger and narrower without changing the total rate appreciably.

3·2. *Asymmetry.* – The asymmetry of the neutrons with respect to the muon spin direction, A of Eq. (5*b*) for right-handed neutrinos (or $-A$ for left-handed neutrinos), is shown in Fig. 2 for the two-component theory and for completely polarized muons, ignoring any pseudoscalar interaction. We note that the fast neutrons display more than $\frac{3}{4}$ of the asymmetry predicted for capture in hydrogen (being $-\frac{1}{3}$ for GT and $+1$ for F) as has already been noted for the case of capture by complex nuclei treated with a Fermi gas model([12]). The asymmetry is somewhat greater when the final state interaction is taken into account; some reason for this will be indicated in Sect. **4**. At low neutron energies for the Fermi interaction the asymmetry has the opposite sign: the reason for this is that there is more phase space available for the final neutrons if the proton before capture is moving parallel to its final direction, and thus the neutron left behind is more likely to be moving oppositely to the fast neutron. For the Gamow-Teller interaction, the ^1S final state predominates so that such an effect cannot occur. In fact, the final state attraction yields such low relative momenta that both neutrons are moving oppositely to the neutrino. We have noted in the introduction that an equal mixture of Fermi and

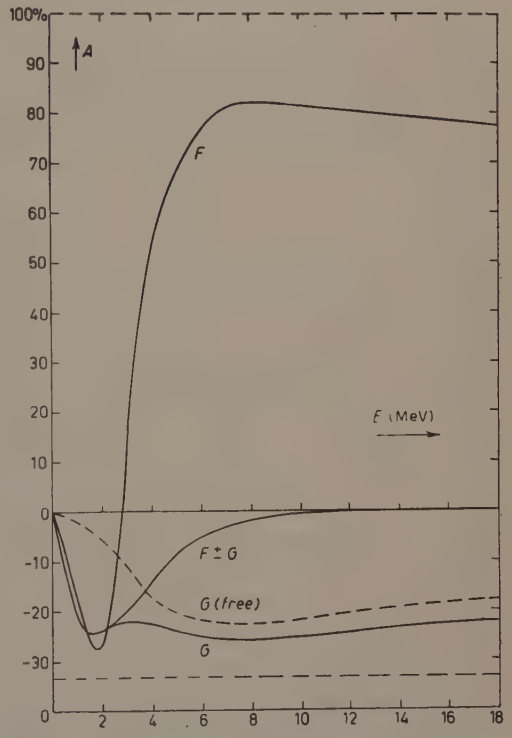

Fig. 2. – Asymmetry parameter A as a function of the neutron energy for pure Fermi and Gamow-Teller transitions and equal mixtures, using two-component theory with right-handed neutrinos (for left-handed neutrinos, the sign of A has to be reversed).

([12]) H. ÜBERALL: *Nuovo Cimento*, **6**, 533 (1957).

Gamow-Teller interaction (called $F \pm G$) gives no asymmetry for capture in hydrogen; we find practically the same result for the fast neutrons for capture in deuterium since the relative enhancement of the Gamow-Teller contribution for deuterium occurs for energies below 5 MeV (Fig. 1). For the case of equal mixture, parity non-conservation displays itself most effectively in the asymmetry of 1 to 3 MeV neutrons, which are nearly all due to the Gamow-Teller interaction.

In Fig. 3 we include the contribution of an effective pseudoscalar coupling $C_P = \pm 9G$ (with $\pm G_F = G_G = G$) as expected from the correction due to virtual pions ([5]). We do not consider here any possible difference between virtual pion effects in hydrogen and deuterium. This pseudoscalar contribution is very important, particularly in the 1 to 3 MeV region, where the Fermi interaction becomes of negligible importance. Thus in this energy region, neutron asymmetries of over 50% are to be expected if C_P/G_G is positive (*) (as expected if $\pi \to \mu + \nu$ is due to the axial vector coupling G_G).

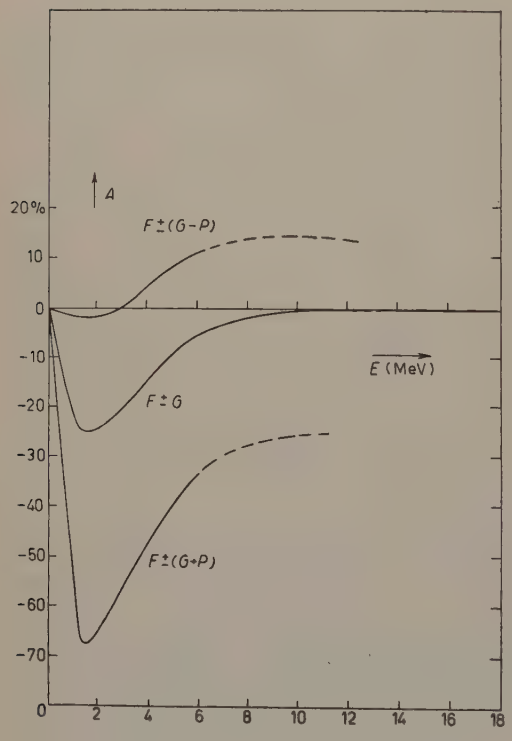

Fig. 3. – Asymmetry parameter A as a function of the neutron energy for equal mixtures of Fermi and Gamow-Teller transitions, plus an effective pseudoscalar interaction of strength $C_P = \pm 9\,G$, for right-handed two-component neutrinos.

3·3. *Neutron polarization.* – The longitudinal polarization for neutrons above 5 MeV is approximately proportional to that for capture in hydrogen ([1,2]), but reduced in magnitude to about 60% of the hydrogen value. For neutrons around 2 MeV the polarization is less than 10% for the cases considered previously; this is due to the overwhelming predominance of the ^1S state (except

(*) The asymmetry is somewhat exaggerated for this case in Fig. 3 due to the neglect of $|C_P|^2$ terms.

for a pure Fermi interaction, in which case there is, however, zero longitudinal polarization for hydrogen), in which the two neutrons must have opposite polarizations.

It is interesting to note that the transverse polarization of the neutrons parallel to the direction given by $\boldsymbol{p} \times \boldsymbol{s}$ does not vanish even if we assume time-reversal invariance. This is the effect of the final state interaction in the ^1S state. A transverse polarization as high as $d = .12$ at 4 MeV and over is possible, being positive if G_F/G_G is positive.

3·4. *Hyperfine structure effects.* – These lead to a difference in capture probabilities for muons in quartet or doublet states of the deuterium mesic atom ([3]). Table I shows the quantity $(\lambda_+ - \lambda_-)/3\bar{\lambda}$ in the case of pure Fermi and Gamow-Teller couplings and of equal mixtures; in the first column for

TABLE I.

	D	D_0
F	0	0
G	-0.876	$-\frac{2}{3}$
$F+G$	-0.628	0
$F-G$	-1.124	$-\frac{4}{3}$

the actual deuteron, in the second column (for purposes of comparison only) for a one-particle model of the deuteron as used by BERNSTEIN *et al.* ([3]) to estimate the effect for a general nucleus. The triplet state suppression due to the exclusion principle is responsible for most of the difference between first and second column.

4. – Capture probability and average asymmetry.

In this Section we consider the total capture probability, which by itself provides a possibility of distinguishing different coupling types. Our results are compared with those obtained using the closure approximation, which is the method usually employed in discussing more complex nuclei. The same approximation is also used to discuss an average asymmetry.

4·1. *Capture probability.* – The capture probability w_D in deuterium has been obtained by numerical integration of the neutron spectrum (Fig. 1) divided by 2. The probability is shown in Table II compared to the decay rate

TABLE II.

	F	G_{Free}	G	$F \pm G$	$F \pm (G+P)$	$F \pm (G-P)$
$10^{-3}(w_{\text{decay}}/w_D)$	48	7.1	5.8	5.1	7.1	4.0
w_D/w_H	0.33	0.75	0.90	0.76	0.71	0.79
w_D/w_H from Eq. (7)	0.39	0.87	0.87	0.75	0.72	0.78

($w_{\text{decay}} = 4.5 \cdot 10^5\ \text{s}^{-1}$) and compared to the capture rate w_H in hydrogen for various assumptions about the coupling (*). An approximate expression for this probability may be obtained if the final nuclear excitation energy (q^2/M) in the energy δ-function of Eq. (4) is replaced by an appropriate constant value, thus determining an average neutrino energy $\bar{\nu}$, and the sum over final states is then performed using the completeness theorem. This approximation should be quite good, since we expect from the neutron spectrum (Fig. 1) that the nuclear excitation energy usually lies between 0 and 10 MeV which corresponds to a variation in ν from about 100 to 90 MeV. This simple closure approximation yields [4]

$$(7) \qquad \begin{cases} w_D = \dfrac{1}{4\pi^2 a_D^3} \dfrac{\bar{\nu}^2}{1 + (\bar{\nu}/2M)} I_D , \\ I_D = a_{FF}[1 - K(\bar{\nu})] + 3[a_{GG} - \tfrac{2}{3} \operatorname{Re} a_{GP}][1 - \tfrac{1}{3} K(\bar{\nu})] , \\ K(\nu) = \int |\psi_d(r)|^2 \cos \boldsymbol{\nu} \cdot \boldsymbol{r}\, d\boldsymbol{r} . \end{cases}$$

The factor $K(\bar{\nu})$ shows the effect of the exclusion principle in the final state in decreasing the transition probability. The values of w_D (divided by w_H) obtained from Eq. (7) choosing $\bar{\nu}$ as 94 MeV (corresponding to q^2/M of about 5 MeV) are also given in Table II. It is seen that this approximation can give

(*) We use the coupling constants $\pm G_F = G$, $G_G = G$, $C_P = \pm 9\ G$, and consistently neglect $|C_P|^2$ terms. $(G = 10^{-5}/M^2)$. With these coupling constants w_{decay}/w_H equals $4 \cdot 10^3$ for the case $F \pm G$ rather than $2 \cdot 10^3$ as has often been stated [13].

([13]) See, for example, J. L. LOPES: *Phys. Rev.*, **109**, 509 (1958). In this case, the difference of a factor 2 is mainly due to Lopes' neglect of corrections of the order μ/M.

errors up to 20%, although it is quite good in some cases. It may be noted that the final state attraction in the ^1S state increases w_D by 20%, whereas Eq. (7) is independent of this interaction (except in so far as it influences the choice of $\bar{\nu}$).

4·2. *Asymmetry.* – The simple closure approximation may also be used to obtain an average value $\langle \boldsymbol{p}\cdot\boldsymbol{s}\rangle$, which is essentially the average neutron asymmetry weighted with the magnitude of the neutron momentum:

$$(8)\qquad I_D\langle \boldsymbol{p}\cdot\boldsymbol{s}\rangle_D = -\tfrac{1}{2}\tfrac{1}{3}\bar{\nu}\left\{b_{FF}[1-K(\bar{\nu})] - [b_{GG}+2\,\mathrm{Re}\,b_{GP}][1-\tfrac{1}{3}K(\bar{\nu})]\right\},$$

which is to be compared to the result for hydrogen

$$\langle \boldsymbol{p}\cdot\boldsymbol{s}\rangle_H = -\frac{1}{3}\nu_H\frac{b_{FF}-(b_{GG}+2\,\mathrm{Re}\,b_{GP})}{a_{FF}+3(a_{GG}-\frac{2}{3}\,\mathrm{Re}\,a_{GP})}.$$

The factor $\frac{1}{3}$ is simply the average of $[(\boldsymbol{\nu}\cdot\boldsymbol{s})/\nu]^2$ over the directions of the neutrino. For a pure Fermi or a pure Gamow-Teller coupling, Eq. (8) gives the simple relation

$$(9)\qquad \langle \boldsymbol{p}\cdot\boldsymbol{s}\rangle_D = \frac{1}{2}\langle \boldsymbol{p}\cdot\boldsymbol{s}\rangle_H\frac{\bar{\nu}}{\nu_H},$$

which has been verified (within better than 5%) for $\bar{\nu}=94$ MeV by the numerical integration of the exact results. This relation indicates that the « average asymmetry » is fairly independent of the nuclear interactions in the initial and final state. Thus the strong final state interaction which has the effect (Fig. 1) of lowering the average neutron momentum also has the effect previously noted of increasing the asymmetry so that $\langle \boldsymbol{p}\cdot\boldsymbol{s}\rangle$ is changed only a little. The significance of the factor $\frac{1}{2}$ may be illustrated by two extreme cases: 1) The neutron and proton are originally separated; then for one-half of the final neutrons, p equals ν_H and the asymmetry is that in hydrogen, while for the other half of the final neutrons, p equals zero. In this case ν equals ν_H and the factor $\frac{1}{2}$ represents the fact that only one-half of the final neutrons come from the capture process, the other half being the « left-behind » neutrons. 2) The two neutrons in the final state stick together so that both neutrons are moving oppositely to the neutrino and both display the same asymmetry as in hydrogen. The factor $\frac{1}{2}$ in this case arises from the fact that p for each neutron equals $\frac{1}{2}\nu$. The double-peaked neutron spectrum for the Fermi case and the single-peaked for the Gamow-Teller show asymmetries qualitatively resembling cases (1) and (2), respectively.

5. – Discussion of approximations.

5·1. *Relativistic effects.* – If we consider the nucleon spin function in Eq. (1) as representing the large components of a Dirac spinor, then the small components have the effect of introducing into $V(i)$ terms proportional to the initial proton and final neutron momenta, $\boldsymbol{p}_{\mathrm{p}}$ and $\boldsymbol{p}_{\mathrm{N}}$. In the closure approximation these terms add to I_{D} in Eq. (7) the term

$$(10) \qquad I_{\mathrm{D}\tau} = 2\lambda \,\mathrm{Re}\,(C_A^* C_G + C_V^* G_F + 2C_T^* G_G) - 4\lambda \,\mathrm{Re}\,(C_V^* G_G)[1 - \tfrac{1}{3}K(\bar{\nu})] - \\ - 2\lambda \,\mathrm{Re}\,(C_T^* G_F)[1 - K(\bar{\nu})] + \lambda^2 |C_P|^2 [1 - \tfrac{1}{3}K(\nu)] ,$$

$$\lambda = \bar{\nu}/2M ,$$

where we have ignored all terms proportional to λ^2 except the term $\lambda^2 |C_P|^2$ and we have given the result explicitly assuming $C' = \pm C$. The terms are of two types, the first three corresponding to terms in $V(i)$ proportional to $\boldsymbol{p}_{\mathrm{N}} + \boldsymbol{p}_{\mathrm{p}}$ $(= \boldsymbol{p}_1 - \boldsymbol{p}_2 = 2\boldsymbol{q})$, and the last three to terms proportional to $\boldsymbol{p}_{\mathrm{N}} - \boldsymbol{p}_{\mathrm{p}}$ $(= -\boldsymbol{\nu})$. Since $\boldsymbol{\nu}$ is fairly constant, the last three terms contribute neutron spectra similar to those in Fig. 1 (the $C_V^* G_G$ and $|C_P|^2$ terms behave like G, while $C_T^* G_F$ behaves like F), and show the effect of the exclusion principle. On the other hand, the first three terms are not decreased by the exclusion principle in this approximation and are relatively more important the higher the neutron energy. In particular for a pure vector interaction, Eq. (10) gives a 30% increase in the total capture rate. For the axial-vector interaction the relativistic effects contribute less than 5% except for neutron energies well above 5 MeV. Eq. (9) remains true for each of these relativistic terms separately, so that the relativistic effects on the asymmetry are similar to those previously discussed for hydrogen [2] except for the relatively greater contribution of the first three terms in Eq. (10).

5·2. *D state of deuteron.* – An estimate of the effect of tensor forces in the initial state has been made, approximating the final two neutron system by an S state only, which should be of sufficient accuracy if there is any important effect. The results are:

a) The rate is changed only by the different normalization of the deuteron S state wave function, which leads to a decrease by a factor $(1 - P_D)$, where P_D is the D-state probability (about 3%);

b) The Gamow-Teller part of the asymmetry is changed from

$$\tfrac{1}{3} b_{GG}(2I'_{tt} + I'_{ss}) \qquad \text{to} \qquad \tfrac{1}{3} b_{GG}(2I'_{tt} + I'_{ss} + 16 \,\mathrm{Re}\,\bar{I}_{ss}) ,$$

with

$$\bar{I}_{ss} = \int_{\nu_{\min}}^{\nu_{\max}} J_s J_s^* \frac{\boldsymbol{\nu} \cdot \boldsymbol{P}}{\nu p} \nu \, \mathrm{d}\nu ,$$

$$\bar{J}_s = -\int (F_s^*)_{S\text{-state}} j_2(\tfrac{1}{2}\nu r) \psi_d^{D\text{-state}}(r) \, \mathrm{d}\boldsymbol{r} .$$

This correction was estimated in the closure approximation and using the asymptotic form of $\psi_d^{D\text{-state}}$; it amounts to only 1% in the neutron asymmetry, which is due to the small overlap of j_2 with $\psi_d^{D\text{-state}}$ rather than to the actual smallness of the D-state radial deuteron wave function, $\psi_d^{D\text{-state}}(r)$.

5·3. *Treatment of the final state interaction.* – This was done for the S-state of the two-neutron system only, and by using the effective-range approximation. The latter is reasonable for up to some $(20 \div 30)$ MeV of relative energy, and can be estimated to hold with sufficient accuracy for neutron energies below $E = (8 \div 10)$ MeV. The error in neglecting P state interactions will probably be small in a similar energy range.

* * *

The numerical calculations were performed on the Paris IBM-704 electronic computer, and the assistance of Miss F. RATTAUD is greatly appreciated.

RIASSUNTO (*)

Si calcolano lo spettro dei muoni, l'asimmetria dei neutroni rispetto alla direzione dello spin dei muoni e la polarizzazione dei neutroni, come funzioni dell'energia dei neutroni per la cattura dei mesoni μ in deuterio. Si danno il rateo totale della cattura e la sua dipendenza dallo stato iperfino dell'atomo mesico e si confrontano con le espressioni approssimate. Si mette in evidenza la dipendenza delle osservazioni dalla quantità di accoppiamento Fermi o Gamow-Teller e dalla presenza di un accoppiamento pseudoscalare effettivo.

(*) *Traduzione a cura della Redazione.*

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Lifetime of Λ^0 θ^0, and Σ^-. (*)

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(ricevuto il 17 Luglio 1958)

Summary. — The lifetimes of the Λ^0, θ^0, and Σ^- observed in bubble chamber photographs have been calculated. The following values are obtained: $\tau_{\Lambda^0}=(2.29^{+.15}_{-.13})\cdot 10^{-10}$ s on the basis of 454 events; $\tau_{\theta^0}=(1.06^{+.08}_{-.06})\cdot 10^{-10}$ s on the basis of 259 events and $\tau_{\Sigma^-}=(1.89^{+.33}_{-.25})\cdot 10^{-10}$ s on the basis of 107 events. A comparison with previous values of the mean lives is presented.

In the course of several exposures with a propane and a hydrogen bubble chamber at the Brookhaven Cosmotron 454 Λ^0's, 259 θ^0's and 107 Σ^-'s were observed. Results will be presented here on the lifetimes of these particles. Preliminary estimates on the same exposures have previously been presented [1].

The hydrogen and propane chambers were cylindrical in shape, 12 in. diameter with the former being 6 in. in depth and the latter 8 in. in depth. They were both placed in a magnetic field of 13.4 kilogauss. The details of

(*) This research is supported by the Atomic Energy Commission and the Office of Naval Research.

(1) F. Eisler, R. Plano, N. Samios, M. Schwartz and J. Steinberger: *Nuovo Cimento*, **5**, 1700 (1957).

the chambers and the methods of analysis are described elsewhere [1]. For the purpose of the lifetime analysis a fiducial region was chosen in the chamber such that Λ^0's, θ^0's or Σ^-'s decaying within this region have track lengths of their decay products sufficient for identification.

For each of the hyperons and the θ^0-meson the actual path, l_i, the potential path, L_i, and the momentum P_i were measured. The potential path is the distance the particle would have travelled in the fiducial region of the chamber had it not decayed. With this information one can calculate for each event a time t_i:

$$t_i = \left(\frac{m}{c}\right)\frac{l_i}{P_i},$$

and a potential time T_i

$$T_i = \left(\frac{m}{c}\right)\frac{L_i}{P_i}.$$

In the above expressions m is the mass of the particle and c is the speed of light. The mass values used in the above calculation were $m_{\Lambda^0} = 1115.2$ MeV [2], $m_{\theta^0} = 494$ MeV [3], $m_{\Sigma^-} = 1196.4$ MeV [3].

In the course of scanning the pictures, events could have been missed due to the following bias. In the case of events in which only one V materializes there is a scanning bias against very short paths of the V. Such events can be interpreted as two prong stars and their omission would increase the value of the mean lives of both Λ^0 and θ^0. Therefore only those single V events were accepted which decayed a distance 3 mm or greater from their origin. For these, 3 mm was subtracted from the actual and potential paths. In cases of double V production the observation of one V enables one to detect the second V to very small distances, so that this fiducial restriction was not applied in that case.

For each particle the likelihood function $L(\tau)$ was calculated:

$$L(\tau) = \prod_{i=1}^{N} \frac{\exp\left[-t_i/\tau\right]}{\tau\left(1-\exp\left[-T_i/\tau\right]\right)}. \quad N \equiv \text{number of events}.$$

The most likely value of τ satisfies the equality

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\log L(\tau) = 0.$$

(2) W. H. BARKAS and A. H. ROSENFELD: UCRL-8030 (1957).
(3) H. H. HECKMAN, F. M. SMITH and W. H. BARKAS: *Nuovo Cimento*, **4**, 51 (1956).

The mean lives thus obtained for the Λ^0, θ^0 and Σ^- are:

$$\tau_{\Lambda^0} = (2.29^{+.15}_{-.13}) \cdot 10^{-10} \text{ s},$$
$$\tau_{\theta^0} = (1.06^{+.08}_{-.06}) \cdot 10^{-10} \text{ s},$$
$$\tau_{\Sigma^-} = (1.89^{+.33}_{-.25}) \cdot 10^{-10} \text{ s}.$$

These errors correspond to one standard deviation in the maximum likelihood method as described by BARTLETT (4).

A possible source of error in the above lifetimes can be due to the misidentification of Λ^0's for θ^0's and vice-versa. At a π^- kinetic energy of 1300 MeV, there were 8 events which were consistent with being either Λ^0 or θ^0. For lower π^- kinetic energies the kinematical separation between Λ^0 and θ^0 was greater so that there were fewer questionable events. All events not clearly identified as one or the other were omitted from the lifetime calculations so that the contamination was at most a few events. The effect of these few events on the lifetime is negligible.

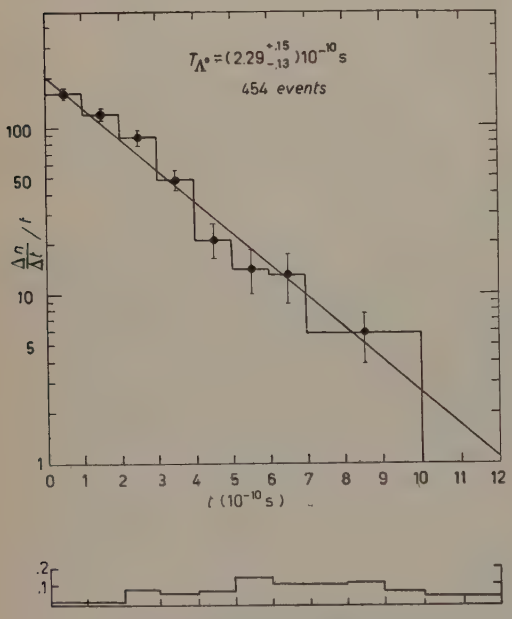

Fig. 1. – Differential decay time distribution for Λ^0. f is the fraction of decays having potential times greater than t. Below is plotted the potential time distribution normalized to unity.

In Σ^- production two charged prongs, one of which shows a break are observed to emerge from a vertex. The contamination in these events which can be due to π^- scatterings has been estimated to be $< 1\%$ (5), so that the error in the Σ^- lifetime due to this effect can be neglected.

The average potential time of each of the particles in the chamber is of the order of three times the observed mean lives. It is therefore possible to draw a differential decay time distribution for the Λ^0, θ^0 and Σ^-. These are shown in Fig. 1, 2 and 3. Also included is the potential path distribution normalized to unity.

(4) M. S. BARTLETT: *Phil. Mag.*, **44**, 249 (1953).
(5) F. EISLER *et al.*: to be published.

In these graphs the number of particles decaying at a time t was divided by the fraction, f, of decays having potential times greater than t. The straight line indicated in each case corresponds to the value of the mean life obtained from the maximum likelihood calculation. In all cases it agrees within a few percent with a least squares fit to the distribution.

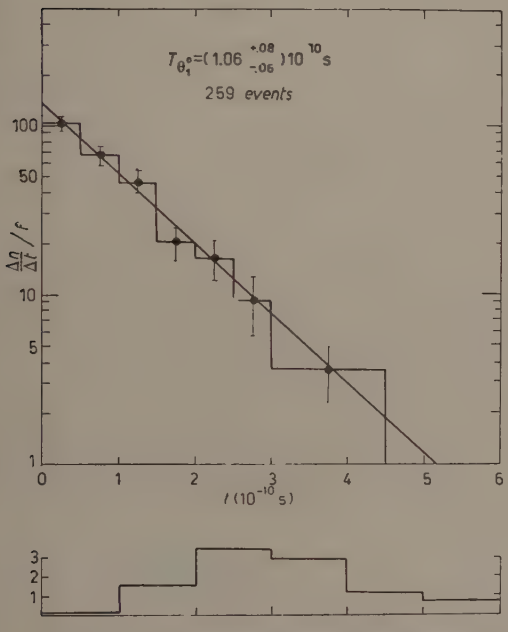

Fig. 2. – Differential decay time distribution for θ^0, similar to that of Λ^0.

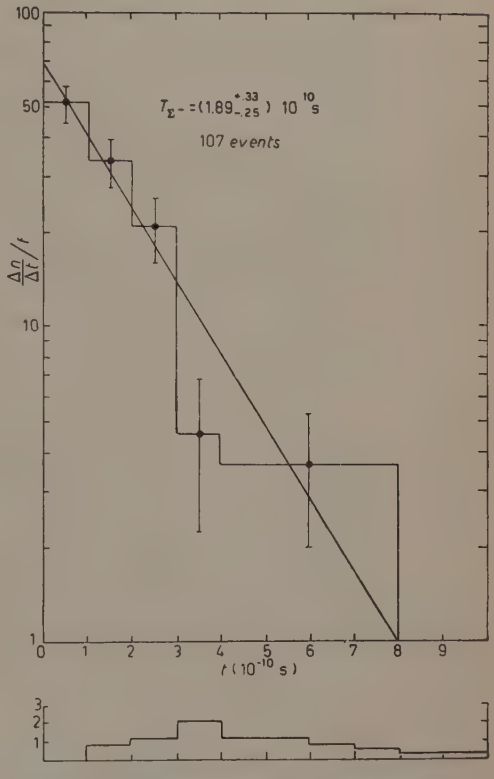

Fig. 3. – Differential time distribution for Σ^-, similar to that of Λ^0.

In Table I we have compiled the latest published values for the lifetimes of the Λ^0, θ^0, Σ^- and Σ^+. The errors quoted correspond to one standard deviation in the Bartlett method [4] except for the results of ALVAREZ *et al.* where the error is one standard deviation for a least squares fit to a differential decay distribution. A weighted average of the various results gives:

$$\tau_{\Lambda^0} = (2.42^{+.13}_{-.12})\cdot 10^{-10}\ \text{s} \qquad \tau_{\theta^0} = (1.05^{+.07}_{-.05})\cdot 10^{-10}\ \text{s}$$

$$\tau_{\Sigma^-} = (1.71^{+.16}_{-.14})\cdot 10^{-10}\ \text{s} \qquad \tau_{\Sigma^+} = (.79 \pm .08)\cdot 10^{-10}\ \text{s}\,.$$

The new Λ^0 lifetime is considerably lower than the earlier Cloud Chamber estimates of $(3.7 \pm .3)\cdot 10^{-10}$ s. The ratio of Σ^- to Σ^+ lifetimes is $2.16 \pm .26$ and agrees with previous estimates.

TABLE I.

Particle	Source	Events	Lifetime (10^{-10} s)	Weighted Average (10^{-10} s)
Λ^0	BLUMENFELD [a]	74	$2.75^{+.45}_{-.38}$	$(2.42^{+.13}_{-.12})$
	ALVAREZ *et al.* [b]	60	$2.95 \pm .40$	
	GLASER *et al.* [c]	29	$2.24^{+.79}_{-.48}$	
	Columbia, Pisa, Bologna	454	$2.29^{+.15}_{-.13}$	
θ^0	BLUMENFELD [a]	39	$1.15^{+.40}_{-.25}$	$(1.05^{+.07}_{-.05})$
	GLASER *et al.* [c]	29	$0.74^{+.21}_{-.15}$	
	Columbia, Pisa, Bologna	259	$1.06^{+.08}_{-.06}$	
Σ^-	ALVAREZ *et al.* [b]	115	$1.6 \pm .2$	$(1.71^{+.16}_{-.14})$
	GLASER *et al.* [c]	44	$1.67^{+.40}_{-.28}$	
	Columbia, Pisa, Bologna	107	$1.89^{+.33}_{-.25}$	
Σ^+	ALVAREZ *et al.* [b]	58	$0.70 \pm .10$	$(.79 \pm .08)$
	SNOW [d]	136	$0.89^{+.14}_{-.10}$	

(*a*) H. BLUMENFELD: *Thesis*, Nevis Report § 37.
(*b*) W. ALVAREZ, H. BRADNER, P. FALK-VAIRANT, J. D. GOW, A. H. ROSENFELD, F. T. SOLMITZ and R. D. TRIPP: UCRL-3775.
(*c*) D. A. GLASER, G. L. BROWN and M. L. PERL: *Phys. Rev.*, **108**, 4, 1936 (1957).
(*d*) G. A. SNOW: *VII Annual Rochester Conference.*

RIASSUNTO

Sono state calcolate le vite medie delle Λ^0, θ^0 e Σ^- osservate in camere a bolle. I valori ottenuti sono i seguenti: $\tau_{\Lambda^0}=(2.29^{+.15}_{-.13})\cdot 10^{-10}$ s, in base a 454 eventi; $\tau_{\theta^0}=(1.06^{+.08}_{-.06})\cdot 10^{-10}$ s in base a 259 eventi; $\tau_{\Sigma^-}=(1.89^{+.33}_{-.25})\cdot 10^{-10}$ s, in base a 107 eventi. Viene presentato un confronto con precedenti valori delle stesse vite medie.

NOTE TECNICHE

A Spectrometer for Protons.

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Istituto Nazionale di Fisica Nucleare - Sezione di Milano

(ricevuto il 5 Luglio 1958)

Summary. — This work describes a spectrometer for protons of energy between 2 and 14 MeV. This spectrometer is particularly suitable for the study of protons emitted by nuclei bombarded with 14 MeV neutrons. The background of this instrument is particularly low. The resolution of the instrument is discussed.

1. – Introduction.

The extended and accurate study of the properties of the particles emitted in nuclear reactions requires the development of new detection methods. Generally several detectors are connected to coincidence and anticoincidence systems, with the aim of selecting the particles that are to be studied and to measuring their properties, such as energy distribution, angular distribution etc.

Recently a special type of proton spectrometer has been studied in this laboratory ([1,2]). It is particularly suitable to detection of protons emitted in the bombardment of nuclei with 14 MeV neutrons.

This paper will discuss the characteristics of a more recent model of proton spectrometer with which it was possible to obtain an appreciable reduction of background pulses with respect to previous spectrometers, and which will permit to extend the study of the angular and energy distribution of the protons emitted with 14 MeV neutrons even to elements with very small cross sections.

([1]) L. Colli and U. Facchini: *Nuovo Cimento*, **4**, 671 (1956).

([2]) L. Colli, U. Facchini, I. Iori, G. Marcazzan, A. M. Sona and M. Pignanelli: *Nuovo Cimento*, **7**, 400 (1958).

We would mention in this connection that for measuring the energy spectrum of protons other types of apparatus have been developed and described in recent publications by C. TRAIL and C. H. JOHNSON [3], by A. GALONSKY and J. JUDISH [4], and by F. L. RIBE and J. D. SEAGRAVE [5].

2. – Description of the Instrument.

The spectrometer concerned is shown in Fig. 1.

It consists of a chamber containing the target of the element under study, three proportional counters, a CsI crystal and a photomultiplier. The object of the proportional counters is to discriminate the protons coming from the target from other particles or from protons emitted by the walls, while the proton energy is measured by means of the scintillation counter.

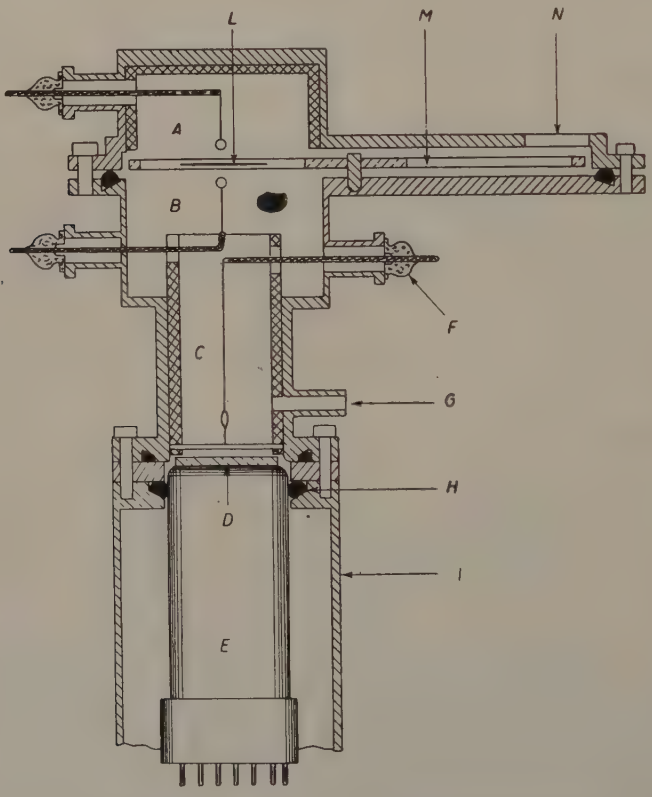

Fig. 1. – The proton spectrometer: *A*, *B*, *C*: proportional counters; *D*: CsI crystal; *E*: photomultiplier; *L*: target; *M*: wheel with target support; *N*: window; *F*: Stapakoff seals; *G*: gas input; *H*: O-ring seals; *I*: container of the photomultiplier.

(3) C. TRAIL and C. H. JOHNSON: *Phys. Rev.*, **95**, 1363 (1954).
(4) A. GALONSKY and J. JUDISH: *Phys. Rev.*, **98**, 1168 (1955).
(5) F. L. RIBE and J. D. SEAGRAVE: *Phys. Rev.*, **94**, 934 (1954).

The target is placed between counters A and B, and only these scintillations are accepted that are produced by particles which have given pulses in B and C but not in A.

The anodes of A and B are wires 20 mm long by 0.2 mm of diameter, while the diameter of the cathode is 70 mm; C is 60 mm long and the diameter of the cathode is 38 mm.

Counter B has a greater diameter than C so that its walls are not visible from the crystal. With such geometry and the use of counter A in anticoincidence, the walls cannot give rise to background pulses.

The cathodes are of graphite so as to have a smaller number of pulses in the individual counters, the wires of A and B are of platinum, and that of C of tungsten.

The common chamber of the three counters is filled with CO_2 at a pressure of 7.5 cm Hg. The biases of counters B and C are so adjusted as to count pulses bigger or equal to those produced by protons having the maximum energy, *i.e.* 14 MeV in the range under study while the bias of A, in anticoincidence, is so adjusted as to count smaller pulses also.

No good proportionality characteristic is required in A and B whereas a fair proportionality is required in C in order to discriminate protons from electrons and to eliminate the pulses caused by α particles by means of a blocking circuit.

The pulses given by α particles of less than $(10 \div 15)$ MeV energy are in effect of higher amplitude than those due to the protons under study.

For its linear response to protons ([6]), a CsI crystal is used, in the form of a disc 1.5 mm thick by 38 mm of diameter. To obtain a stable optical contact, even in vacuum, the crystal was cemented to the head of a Dumont 6292 phototube. To screen the crystal from the light produced by the discharges of the counters and to improve the light collection (of the scintillations) an aluminium foil of 0.02 mg/cm^2 is placed on the crystal.

The targets are carried by a wheel inside the chamber, that can be turned from the outside by means of a magnet so as to enable the replacement of the targets without opening the counter.

Across the four holes in the wheel are stretched two thin tungsten wires, by means of which the targets are supended. It is all so arranged that only the target and part of the wires, but not the wheel, are visible from the crystal. A hole is left free for the measurement of the background.

The pulses given by the photomultiplier in coincidence with the counters B and C, and not blocked by A, are sent to a 99-channel pulse analyzer.

The resolution of the coincidence circuits has been fixed at 5 μs taking into account the rise time of CsI.

The flux of the incident neutrons is so regulated that casual coincidences can be regarded as negligible. Under these conditions a flux of the order of 10^6 incident neutrons per second on the target is convenient.

The calibration of the energy scale was done by means of recoil protons produced in a target of polythene of 13 mg/cm^2 thickness and 1 cm diameter, placed at a distance of 100 mm from the neutron source.

The spectra obtained in these conditions are shown in Fig. 2. The energy

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([6]) S. Bashkin, R. R. Carlson, R. A. Douglas and J. A. Jacobs: *Phys. Rev.*, **109**, 434 (1958).

of this proton line is determined by the energy of the incident neutrons and the emission angle of the protons, taking into account the energy loss in the target and the gas. Since the CsI crystal is known to be linear to the protons. the entire energy scale can be obtained.

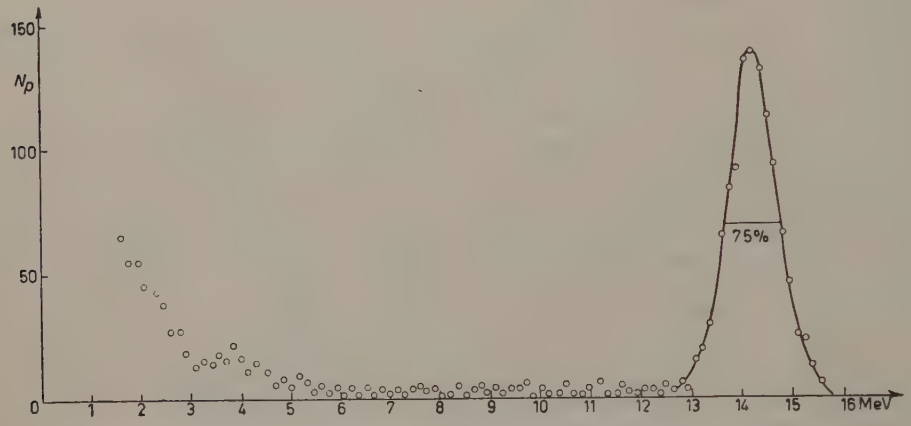

Fig. 2. – Spectrum of proton recoils due to 14.5 MeV neutrons.

In order to evaluate the effect produced by n, p reactions with respect to the background of the instrument Fig. 3 shows a spectrum produced by an Al target of 21 mg/cm² thickness and the relative background. By using

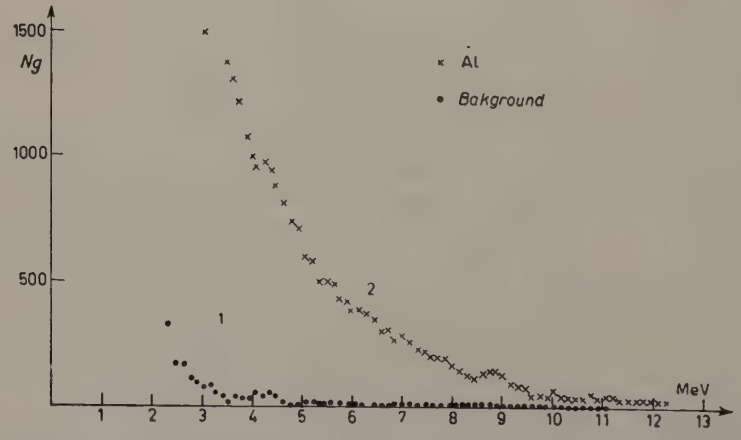

Fig. 3. – Background (1) and proton spectrum of Al(n, p) given by target 21 mg/cm² thick (2). The energy scale was corrected for energy loss in the target and gas.

a target of 50 mm diameter, at a distance of 100 mm from the neutron source the emission angle of protons shows a spread of about 20°.

3. – Discussion.

It is interesting to discuss briefly the energetic resolution of the instrument. Assuming a monoenergetic proton group, the width of the proton line is determined mainly by the crystal-photomultiplier assembly and by the thickness of the target. In fact the width due to the whole amplification and coincidence chain can be made quite small. So it is possible to write:

$$\Lambda = \sqrt{\lambda_{\text{th}}^2 + \lambda_{\text{ph}}^2}\,.$$

The term λ_{ph} depends on known factors as the statistical fluctuations in the light yield by the crystal, optical collection of the light and fluctuation in the production and collection of photoelectrons and in the multiplication process. This term can be affected even by the inhomogeneithy of the crystal and the disuniformity of the photocathode.

When the crystal is mounted as in Fig. 1, directly on the photocathode, we have a superposition of these two kinds of disuniformity.

A way for avoiding the effect of photocathode inhomogeneity is to distribute the light from any point of the crystal uniformly on the photomultiplier by means of a light guide.

In fact using this method it was possible to obtain some small improvement on the resolution.

With these conditions, from the different resolutions obtained using the whole area or a small region of the crystal it is possible to deduce that the term λ_{ph}, for a crystal of (35÷40) mm diameter, is principally due to crystal inohomogeneity. In such a case λ_{ph}, it is not strongly depending on the proton energy.

The term λ_{ph}, that represents the instrumental resolution, in our apparatus, at 14 MeV results of the order of 4%.

The other factor affecting the resolution, in the energy spectra, is the target thickness. A good situation is to use a target thickness giving a λ_{th} of the same order of λ_{ph}.

But λ_{th} is strongly depending on the energy of the emitted particles. With the usual targets, this term is the most important in the low energy region, where the limits are due to the background combined with the resolution.

With a target of Al, 4 mg/cm² thick, (that is a good thickness with reference to the background of our spectrometer) λ_{th}, at 2 MeV, results 28%.

* * *

We gladly thank dr. F. Chaminade for private communication and prof. U. Facchini and prof. L. Colli for their helpful discussions and advice.

RIASSUNTO

In questo lavoro è descritto uno spettrometro per protoni di energia tra 2 e 14 MeV. Questo spettrometro, progettato per lo studio di protoni emessi da nuclei bombardati con neutroni di 14 MeV, presenta un fondo particolarmente basso. Viene discussa inoltre la risoluzione dello strumento.

Optical-Electronic System for Transmitting the Data of an Electric Measuring Instrument.

D. Brini and L. Peli

Istituto di Fisica dell'Università - Bologna

(ricevuto il 10 Luglio 1958)

Summary. — Description of a device transmitting, through light pulses, the data of an electric measuring instrument placed in a high voltage container. Its essential feature is its high stability, obtained by eliminating the dependence of the oscillator frequency from the characteristics of the oscillator tube.

It is often necessary to transmit the data of a measuring instrument far away from where it stands without using cables. This is the problem to be solved in the case of linear accelerators of particles, where functioning conditions of parts at high voltage within the accelerator, ought to be known.

Usually the transmission of these data is done optically, using the light of an adequate source. For such transmission two methods are followed:

1) through a suitable device, a correlation between the measured voltage and the light intensity emitted by a lamp is established. This light is collected and re-transformed in voltage or current varying according to the intensity of the light itself. This method is seldom used owing to its difficulties: first of all the relation voltage-light intensity is difficult to be kept constant for a long time because of the illuminating lamp; in addition a good long-term stability of the gain of the photomultiplier collecting the light is required;

2) through a convenient device a correlation is made between measured voltage and the frequency of an oscillator monitoring a lamp emitting light pulses. The light pulses are observed by means of a photomultiplier and are transformed in voltage pulses whose frequency is measured (1).

(1) G. Cortellessa and R. Querzoli: *Rend. Ist. Sup. Sanità*, **17**, 359 (1954); *Nuovo Cimento*, **11**, 321 (1954).

This second method is more frequently used because it eliminates or reduces the difficulties of the first one. The most important difficulties are in the correlation voltage-frequency of the oscillator depending not only on the passive components used in the circuit, but also on the characteristics of the tube.

Our apparatus follows this second method but in it the correlation voltage-frequency of the oscillator is made independent of the characteristics of the oscillator tube, being therefore only influenced by the passive components which, if carefully chosen, have a good stability and long duration.

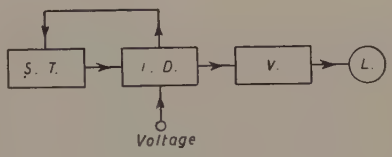

Fig. 1.

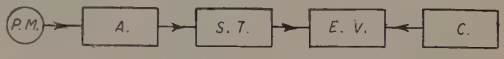

Fig. 2.

Fig. 1 shows the block diagram of our transmitting apparatus. One part of the voltage to be measured is taken to give the bias to a conventional integral discriminator (I.D.). This is triggered by the properly linearized saw teeth, coming out from a thyratron (S.T.). The output of the integral discriminator triggers in its turn (S.T.). So, once the rise time of the linear saw tooth is fixed, using voltage stabilized supply, the frequency of the saw

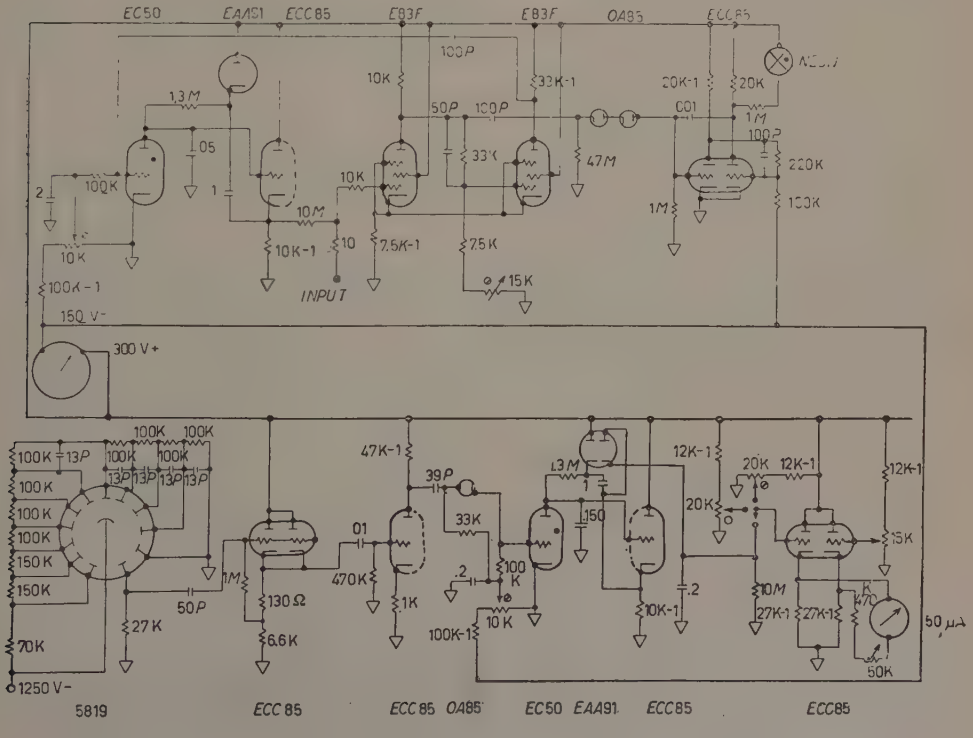

Fig. 3.

teeth does not depend on the characteristics of the oscillator tube. (I.D.) triggers a univibrator (V) which lights the lamp. The lamp (L) used is a normal neon lamp, the maximum frequencies being of the order of magnitude of 100 Hz.

Fig. 2 shows the block diagram of the receiving apparatus. (P.M.) is a photomultiplier collecting the light pulses ((P.M.) don't need to have an especially high stability in gain). The tension pulses coming from (P.M.) are amplified by a single amplification stage (A) whose output triggers a saw teeth generator (S.T.); this is a tyratron with a linearity circuit. (E.V.) measures the height of the saw teeth coming out from (S.T.) after suitable integration. The saw teeth being linear, their height is proportional to the voltage to be measured. The rise time of these saw teeth need not be equal to the rise time of the saw teeth of the transmitter: only the condition of linearity is to be maintained. (C) is a device which allows to control and correct any change of zero and of the circuit sensitivity. Tests of linearity and stability have been made for a long time.

Fig. 3 shows the circuit constructed and mounted on the 600 keV Cockroft-Walton accelerator of the Istituto di Fisica - Università di Bologna.

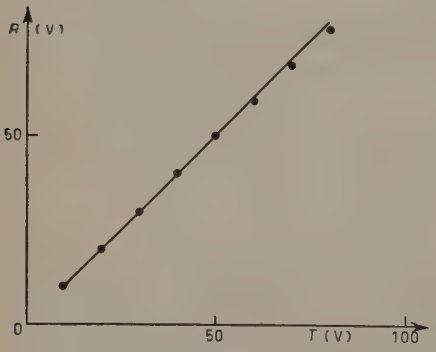

Fig. 4.

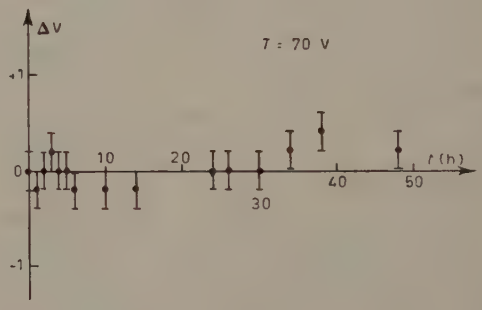

Fig. 5.

Fig. 4 shows the behaviour of the measured voltage in the receiver (R) compared to the voltage measured in the trasmitter (T.) The linearity is very good, remaining within 1%, being therefore consistent with the linearity of the electromagnetic instruments used.

Fig. 5 shows the difference (v. time) between the voltages of the receiver and trasmitter. It can be observed that the stability is contained in a fraction of 1%, as foreseen.

* * *

We wish to thank Mr. F. MARANI for his help in assembling and setting the device.

RIASSUNTO

Descrizione di un dispositivo trasmittente, attraverso impulsi luminosi, i dati di uno strumento elettrico di misura posto in un contenitore ad alta tensione. Il pregio fondamentale è la sua alta stabilità, ottenuta eliminando la dipendenza della frequenza dell'oscillatore dalle caratteristiche della valvola oscillatrice.

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Preliminary Results on K^+ Interaction at High Energy (200 ÷ 350 MeV) (*).

M. Grilli, L. Guerriero, M. Merlin (†), Z. O'Friel (×) and G. A. Salandin

Istituto di Fisica dell'Università - Padova
Istituto Nazionale di Fisica Nucleare - Sezione di Padova

(ricevuto il 16 Giugno 1958)

Technical details.

A stack of 240 Ilford G-5 pellicles (20 cm × 17.5 cm × 14.5 cm) has been exposed to the doubly focussed K^+-beam ($p = 625$ MeV/c) at the Berkeley Bevatron and shared among the laboratories of Bristol, Dublin U.C. and Padova (†).

In order to pick up the tracks to be followed, the ionization of all tracks parallel to the beam within 5° has been measured. The values appear to be grouped in two peaks at $I/I_0 = 1.0 \pm 0.1$ (corresponding to the π^+) and $I/I_0 = 1.3 \pm 0.1$ (corresponding to the K^+).

Every K-track has been followed for ~ 9 cm, and the interactions have been recorded in the energy interval 200 ÷ 350 MeV. The nature and energy of the interacting particles have been determined by $g - \bar{\alpha}$ measurements. From these measurements and from $\Delta g/\Delta R$ measurements over a sample of tracks followed for ~ 20 cm, it was possible to determine the contamination of the selected tracks and the entrance energy of the K^+'s. The contamination was found to be negligible (< 2%) and the entrance energy showed a flat distribution between 260 and 340 MeV.

Results.

Table I shows a summary of the events found on a total track length of 60.5 m.

We have accepted as K^+-H interactions all events satisfying the energy-momentum conservation; as an indication of their reliability we quote transversal momentum p_T normalized respectively to the primary momentum p_1, and

(*) Per una svista tipografica, la presente, non ancora licenziata, fu pubblicata priva delle figure nel fascicolo del 16-7-1958. La si ripubblica ora in forma definitiva.

(†) At present at the Bari University.

(×) Fulbright Research scholar on leave from St. Bonaventure University, Olean, N.Y., U.S.A. Z. O'Friel wishes to express his thanks to the United States Government for a Fulbright grant during the time of this research and gratefully acknowledge the kind hospitality of his colleagues for the use of their laboratories, especially Prof. N. Dallaporta and Prof. A. Rostagni, Director of the Institute of Physics, Padova.

(†) We are very grateful to Prof. J. E. Lofgren and his collaborators for the exposure of the stack. Our thanks are also due to the Bristol group for the processing of the plates and to the colleagues of Bristol, Dublin U.C., U.C.L.A. and B.N.L. who kindly supplied their data previous to publication.

TABLE I.

Type	Scatterings ($\theta_{Lab} > 20°$)			Charge exchanges			Not analyzable	K-H	Decays in flight	Track followed (meters)
	a (*)	b (*)	Total	a (*)	b (*)	Total				
No. of events	16	52	68	1	28	29	8	7	8	60.5

(*) a = Events with no prongs (excluding the recoil),
b = Events with prongs (excluding the recoil).

The details on K^+-H events are as follows:

No. of event	E_1 (MeV)	θ_{CM}	p_T/p_1	p_T/σ
K_{IV} 678	230	76	0.026	0.47
620	232	135	0.47	1.14
99	250	38	0.021	1.20
192	260	94	0.017	0.33
637	265	116	0.008	0.24
386	270	106	0.017	0.45
802	325	44	0.019	0.43

to p_T standard deviation (σ) (evaluated for each event) (+).

Table II shows the values of the mean free path for inelastic scattering (charge-exchange included).

The ratio charge exchanges/scattering turns out to be .43 ± .10. In Fig. 1 are plotted the values of $\Delta E/E_1$ *vs.* θ_{Lab} for all scatterings. The curve reported

TABLE II.

Λ (emulsion) cm	Λ_1 (emulsion (*) cm	λ (nuclear matter) Fermis
57 ± 6	55	6

(*) Λ_1 is the m.f.p. corrected for Coulomb effect.

(+) We have calculated the p_T distribution to be expected for K-bound proton collision (Fermi model; $p_{max} = 241$ MeV/c). From this distribution and from the total number of two prong scatterings, we have calculated that the contamination in our K-H sample is less than 0.5 event. Further details and results will be given at the Geneva Conference.

in Fig. 1 corresponds to the K-free nucleon collisions. Fig. 2, 3 show the energy loss and the angular distributions, res-

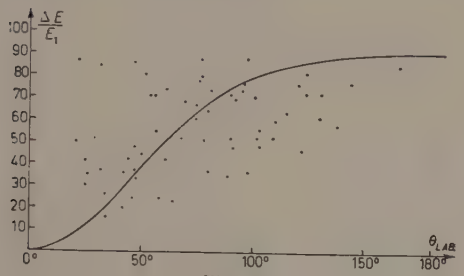

Fig. 1.

pectively. We report in Fig. 4 the prong distributions of the scattering (continuous line) and of charge exchanges (dotted line).

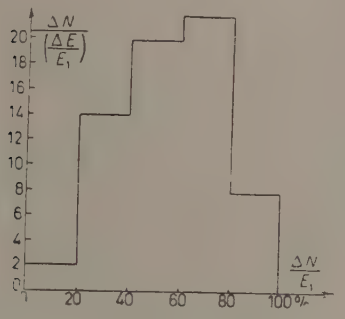

Fig. 2.

We have found a one-prong star, in wich a positive pion has been produced by collision of a K with a nucleus. The

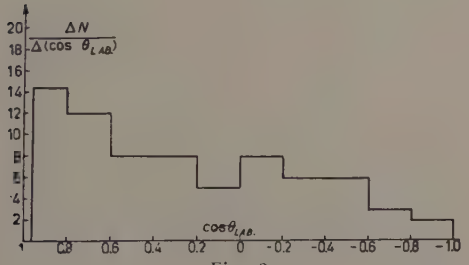

Fig. 3.

nature and the energy of the primary K has been determined by accurate measurements of $\bar{\alpha} - g$. The nature of the prong comes out from the typical $\pi^+ \to \mu^+ \to e^+$ decay at rest. The energy of the π has been determined by its range.

The main data are:

Particle	Energy (MeV)	θ_{Lab}	Remarks
K	300 ± 24	—	—
π	46	89	π-μ-e at rest $R = 30.6$ mm

We can exclude this event to be a decay in flight of a K. In fact, if we should interpret this event as a decay

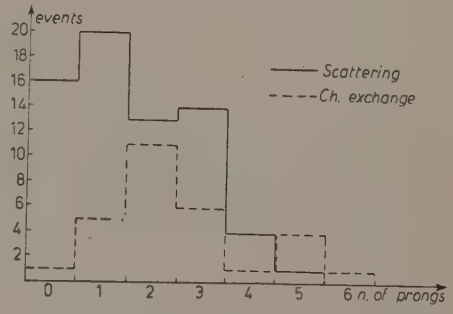

Fig. 4.

in flight, the energy of the π in center of mass system should be 165 ± 15 MeV (*i.e.* clearly greater than the energy of a π from a $K_{\pi 2}$ decay).

The elementary reaction:

$$K^+ + n = K^0 + n + \pi^+$$

is energetically possible.

Discussion.

1) Our data on K-H collisions seem to indicates an increased σ_{Kp} with respect to the results at lower energy ($\sigma = (14.5 \pm 2.2)$ mb for $E_1 < 220$ MeV). By adding the data of other labora-

tories (1) we obtain $\sigma_{Kp}=(24^{+8}_{-7})$ mb (E_1 between 200 and 350 MeV).

2) The quoted value of $\sigma_{K\text{-nucleus}}$ for inelastic events is in agreement with that found by other authors (1). From all these values a clear trend appears of the $\sigma_{K\text{-nucleus}}$ to increase also for $E_1 > 200$ MeV, as for the lower energy (2,3).

The value of the m.f.p. (λ) in nuclear matter at $E_1 \sim 125$ MeV enables us to evaluate the probability of double scatterings of a K^+ inside one nucleus. This turns out to be $\sim 32\%$ (assuming $E_1 = 250$ MeV and $\Delta E/E_1 = 50\%$ in the first collision). Due to such an high frequency of double scatterings the interpretation of $(d\sigma/d\Omega)_{Lab}$ (Fig. 3), *i.e.* the valuation of the $(d\sigma/d\Omega)_{CM}$ for K-nucleon collision, becomes rather complicated. Some events with large $\Delta E/E_1$ are inconsistent with a single collision K-nucleon within the nucleus (Fig. 1). These events are attributable to double scatterings of K^+ within the same nucleus. It seems unlikely that such events are due to pion production and reabsorption, owing to the small cross-section of the pion production reaction.

3) In Fig. 5 we report a summary of the present data on the ratio charge exchanges scattering. The relative frequency of charge exchanges to the inelastic events for $E_1 > 200$ MeV is clearly greater than at lower energies. To obtain the true (*i.e.* in the interaction K-free nucleon) variation of the ratio C.E./scattering with E_1, corrections should be applied to the experimental values to account for the effect of Pauli principle and of double scatterings. We point out that the occurrence of double scatterings at high energy leads to an

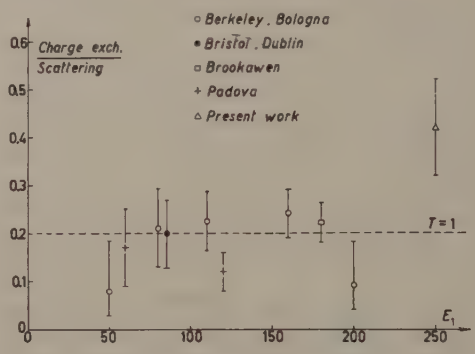

Fig. 5.

increase of the experimental relative frequency of charge exchange to scattering events. At $E_1 = (200 \div 350)$ MeV ($\sim 32\%$ of double scatterings) a true ratio C.E./scatt. = 0.20 (pure $T=1$ state) would give an experimental ratio 0.27. The correction to this ratio at these energies arising from the Pauli principle is less than 0.03. The observed value C.E./scatt. = 0.43 (*) shows therefore a clear contribution of a $T=0$ state.

In a previous analysis of inelastic scattering of K^+ we found the presence of $T=0$ state also for $E_1 = (80 \div 150)$ MeV, (3) with a true ratio C.E./scatt. $= 0.28 \pm 0.10$. The quoted value 0.43 seems to indicate that the contribution of the $T=0$ state is more rapidly increasing with the energy than the contribution of the $T=1$ state.

These results could be explained assuming that the K-nucleon potential is attractive in the $T=0$ state, and repulsive in the $T=1$ state, in agreement with the signs of the scattering amplitudes evaluated by us (3) for $E_1 = (80 \div 150)$ MeV ($a_{10} < 0$; $a_{01} > 0$). If it is so, the $T=0$ state should give a resonance at higher energies.

(1) Communications of the plate groups of Bristol, Los Angeles (U.C.L.A.), and Padua laboratories at Washington meeting (May 1958), reported by Dr. D. J. PROWSE.

(2) See Fig. 1 in B. SECHI-ZORN and G. T. ZORN: *Phys. Rev.*, **108**, 1098 (1957).

(3) See Fig. 4 in M. GRILLI, L. GUERRIERO, M. MERLIN and G. A. SALANDIN: *Analysis of inelastic interactions of* K^+*-mesons with emulsion nuclei* (40 ÷ 150 MeV) (circulated preprint, submitted to *Nuovo Cimento*).

(*) The value reported at Washinghton Meeting (1) for this ratio is 0.42 ± 0.05.

4) Further support for the reliability of the elementary reactions:

(I) $K^+ + p = K^+ + p$,

(II) $K^+ + n = K^+ + n$,

(III) $K^+ + n = K^0 + p$,

may be obtained from the following considerations.

a) In five charge exchange events the dynamics of the collision (angle and energy of emitted proton) is consistent, within the errors, with a K-n collision (the neutron being pratically at rest).

b) The average number of prongs is greater for charge exchange events ($n=2.6$), than for scatterings ($n=1.6$), (see Fig. 4) as it is to be expected on the basis of the reactions (I), (II) and (III).

Starting from these values of n it is also possible to evaluate roughly the frequency of reactions (I) and (II). We found that the contribution of the reaction (II) is certainly at least comparable with that of the reaction (I) at $E_1 > > 200$ MeV.

5) Until now three cases [4,5] have been reported of charged π produced in K^+-nucleus collisions, on a total of ~ 500 inelastic events. To evaluate the frequency of π production in K-nucleon collisions we should take in to account:

a) Possible π^0-production. This reaction would give rise to events with a very large $\Delta E/E_1$. In fact, such events have been observed (see Fig. 1). In our cases, however, the emission of a π^0 is to be excluded by energy considerations.

b) Production and subsequent absorption of a π in the same nucleus, charge exchanges of charged π. The evaluation of these effects is now in progress.

Neglecting these corrections, the relative frequency

$$\frac{\pi\text{-production}}{\text{Inelastic events}}$$

can be roughly evaluated and results to be ~ 0.006. This value is larger than predicted by preliminary theoretical calculations, assuming either a direct π production by nucleon via a KNY interaction or a $KK\pi\pi$ interaction (*).

(4) B. SECHI-ZORN and G. T. ZORN: *The production of a π^--Meson by a K^+-particle* (circulated preprint, October 1957).

(5) E. HELMY, J. H. MULVEY, D. J. PROWSE and D. M. STORK: *An example of the production of a π^--Meson by a K^+-Meson* (circulated preprint, April 1958).

(*) C. CEOLIN, N. DALLAPORTA and L. TAFFARA: *Nuovo Cimento*, **10**, 186 (1958). We are very grateful to these colleagues for many illuminating discussions on this subject.

On the Scattering of (200÷300) MeV K^+-Mesons (*).

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(ricevuto il 16 Luglio 1958)

1 – Introduction.

Earlier work on the scattering of K^+-mesons of energies between 40 and 200 MeV in nuclear emulsion has shown [1-7] that the scattering is mainly due to the $T=1$ isotopic spin state and that it is primarily an s-state interaction. The contribution of the $T=0$ state is small and to account for the backward peaking of the K^+-nucleon differential cross-section is usually considered to be p-wave. The $T=0$, s-state is not believed to be significant.

Preliminary results have now been obtained at a higher energy and the nature of the scattering appears to have changed considerably. Similar results are presented by the University of Padua group in an adjoining communication.

(*) Part of this work was reported together with some of the data from the Padua group at te Washington Meeting of the American Phyhsical Society, May 1958.

(−) On leave from the University of Panjab. India.

(+) On leave from the University of Aligarh, India.

(†) Now at Clarendon Laboratory, Oxford.

(×) On leave from the University of Bristol.

(°) Partially supported by the United States Atomic Energy Commission.

(1) S. Biswas, L. Ceccarelli-Fabbrichesi, M. Ceccarelli, K. Gottstein, N. Varshneya and P. Waloschek: *Nuovo Cimento*, **5**, 125 (1957).

(2) G. Cocconi, G. Puppi, G. Quareni and A. Stanghellini: *Nuovo Cimento*, **5**, 172 (1957).

(3) M. Baldo-Ceolin, M. Cresti, N. Dallaporta, M. Grilli, L. Guerriero, M. Merlin, G. Salandin and G. Zago: *Nuovo Cimento*, **5**, 402 (1957).

(4) B. B. Bhowmik, D. Evans, S. Nillson, D. J. Prowse, F. Anderson, D. Keefe, A. Kernan and J. Losty: *Nuovo Cimento*, **6**, 440 (1957).

(5) T. F. Hoang, M. F. Kaplon and R. Cester: *Phys. Rev.*, **107**, 1698 (1957).

(6) M. Widgoff, A. Pevsner, D. F. Davis, D. Ritson, R. Schluter and V. P. Henri: *Phys. Rev.*, **107**, 1430 (1957).

(7) J. E. Lannutti, S. Goldhaber, G. Goldhaber, W. W. Chupp, S. Giambuzzi, C. Marchi, G. Quareni and A. Wataghin: *Phys. Rev.*, **109**, 2121 (1958).

2. – Exposure.

The emulsion stacks were exposed to the separated K^+-meson beam at the Berkeley Bevatron, the nominal momentum of which was 625 MeV/c. The K-mesons passed through 26 in. of beryllium degrader before entering the stacks and had lived for a time of $2 \cdot 10^{-8}$ s.

The U.C.L.A. stack was large enough to arrest those K^+-mesons which did not interact in the emulsion. A range histogram of a random sample of K-mesons followed to rest is shown in Fig. 1. Using the Barkas range-energy

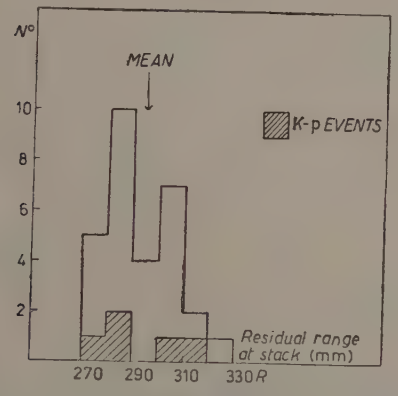

Fig. 1. – Residual range histogram of the non-interacting K^+-mesons followed to rest. The ranges are measured from the front edge of the stack (there was no intervening wrapping of any kind). Five scattering events from free protons have been observed in which both particles were arrested, the total energy of these particles provides an independant determination of the beam energy. The equivalent residual range of the K^+-mesons for these five events are shown cross-hatched. An additional check is provided by a $K_{\pi 2}$-decay in flight in which the secondary π-meson has been arrested; this event gives a primary K-meson energy of (304±10) MeV, in excellent agreement with the determination from the mean residual range.

relation ([8]), we obtain from the mean residual range of (295±5) mm a mean beam energy of (304±3) MeV, the spread being about 10 MeV.

([8]) W. H. BARKAS: UCRL-2426.

3. – Experimental Details.

A total of 90 m of K^+-meson track has been followed to date between the limits 200 and 300 MeV. The tracks are picked up 15 mm from the leading edge of the stack and in general a count of 400 blobs is made on every track visible. A track is followed if the blob-density lies within certain limits, usually for a range of 10 cm, or until it decays in flight or interacts either elastically or inelastically. A second count is made at the end of the 10 cm range to exclude background tracks of the same initial blob density as the K-mesons.

The limits imposed for the Bristol sample allowed some small contamination of background tracks but this was corrected for by estimating the proton and π-meson track length followed from the number of such interactions found.

All decays in flight, inelastic events and elastic events in which the projected angle of scatter was $\geqslant 5°$ were noted and all « elastic » events in which the angle of scatter was $\geqslant 20°$ were examined for inelasticity. The identity of all outgoing tracks from the interactions has been determined and in events in which no K-meson was emitted we are confident of the identity of the K-meson primary. On a sample of track length followed all elastic scattering events in which the projected angle was $\geqslant 2°$ were also noted and space angles measured. Care was taken to correctly identify all K-p elastic events and to identify any π-mesons emitted in the interactions.

4. – Results.

The results are summarized in Table I. The mean free path in nuclear emulsion is seen to be (53.9±4.2) cm, the rapid rise in the cross-section which led to a rapid fall in the m.f.p. between 50 and

TABLE I. – *Statistical summary.*
Mean beam energy at stack entrance (304±3) MeV. *Details of interactions.*

	Metres followed	K-p Events	Charge exchange events			Inelastic events			Not analysable	Total
			Stars	Stops	Total	Stars	0-prong	Total		
Bristol	41 (*)	0	16	3	19	50		50	10	79
UCLA	49	5	26	2	28	49	9	58	2	88
Total	90	5	42	5	47			108 (+)	12	167

1 interaction producing a π^--meson has been observed ([14]) and probably 1 producing a π^0.

M.F.P. (all events excl. K-p and π-production) = 53.9±4.2 cm.

Charge exchange/Total ratio observed = .303±.05, corrected for double scattering = 0.25±.04.

K-p cross-section including 7 Padua events and 1 Dublin event reported at the Washington meeting of the A.P.S., from a total meterage of 143 m = (21.6±6.5) mb (×).

(*) Corrected for background tracks followed.

(+) Only 106 of these events are plotted in Figs. 2, 3 and 4.

(×) We are very grateful for prior knowledge of the Padua and Dublin events and we wish to thank these groups for their private communications.

200 MeV has thus flattened out as this value is in agreement with the results at 200 MeV ([7,9]). The K-p cross-section of ∼13 mb (*) from only five events is in agreement with the values obtained at lower energies. The details of the five events are shown in Table II. The ratio of the number of charge-exchange events to the total is 0.30±.05. This is considerably higher than the figures close to the limit for a vanishingly small interaction in the $T=0$ state, 0.167, obtained at lower energies. The m.f.p. of 125 MeV K^+-mesons (average energy after the first scatter) of ∼ 70 cm ([7]) implies that about 40% of events will be the result of double and not single scattering within the nucleus. When this effect has been corrected for the ratio of C.E. to total at the first scatter becomes 0.25±.04. This is in good agreement with the results of ZORN and SECHI-ZORN ([9]) at a similar energy.

TABLE II. – K-p *events.*

Event	E_K	θ	$\varphi_{C.M.}$	$\cos\varphi_{C.M.}$
J_{10}	261	132	158	−.93
J_{30}	248	4.2	8.8	+.99
J_{50}	239	137	161	−.96
P_{22}	238	66	102	−.21
J_5	226	57	90	0

(*) A better figure for the K-p cross-section was reported at the Washington Meeting of the American Physical Society (May 1958), which is quoted in Table I.

([9]) G. T. ZORN and B. SECHI-ZORN: *Phys. Rev.*, **108**, 1098 (1957) and *Bull. Am. Phys. Soc.*, II, **3**, n. 1, 24 (1958).

The K-nucleon differential cross-sections in the laboratory and centre of mass systems are plotted in Figs. 2 and 3. A free nucleon conversion has been used to obtain the C.M. cross-sections and

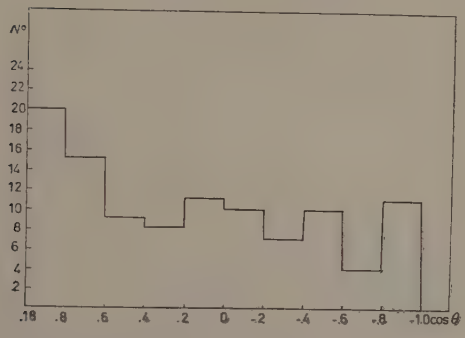

Fig. 2. – The angular distribution of the K⁺-mesons re-emitted from nuclear interactions plotted in the laboratory system.

double scattering has been neglected. Double scattering will be expected to slightly peak the true distribution backwards.

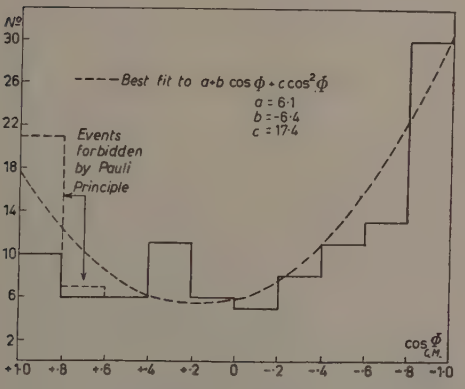

Fig. 3. – The angular distribution of K⁺-nucleon scattering plotted in the C.M. system.

In addition, the Pauli exclusion principle will strongly inhibit events above $\cos\varphi = 0.8$. Taking the latter into account as shown in the figure, the distribution has a fairly clear $a + b\cos\varphi + c\cos^2\varphi$ appearance, with the coefficients as listed. The actual values should not be taken too seriously but they clearly indicate appreciable p-wave interaction which is probably of opposite sign to that of the s-wave.

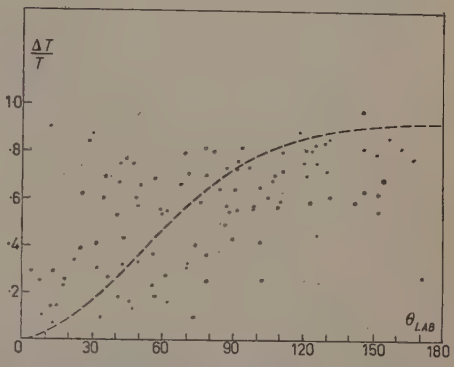

Fig. 4. – Distribution of $\Delta T/T$, the fractional energy loss, of the K⁺-mesons emitted in the interaction.

The distribution of $\Delta T/T$, the fractional energy loss of the K⁺-mesons emitted in the interactions, is shown plotted in Fig. 4 against θ, the laboratory angle of scatter. There is an excess of events below the line representing free K - meson - nucleon scattering. Double scattering will tend to increase $\Delta T/T$, the real excess below the line for single scattering is thus increased. This excess indicates that the potential within the nucleus which was shown to be repulsive at lower energies ([1-7]) is still repulsive at 250 MeV.

The elastic scattering of K-mesons from emulsion nuclei can be used to obtain a more exact figure for this repulsive potential. 49 m of track have been scanned for elastic events. The differential cross-section per emulsion nucleus is shown in Fig. 5. A preliminary analysis of this curve has been made on the basis of the optical model using a Saxon ([10]) type potential. The angular distributions have been calcu-

([10]) M. A. MELKANOFF, N. NODVIK, D L. S. SAXON and R. WOODS: *Phys. Rev.*, **106**, 793 (1957).

lated exactly using a modified code prepared by Dr. M. A. MELKANOFF and Dr. D. S. SAXON, originally used in

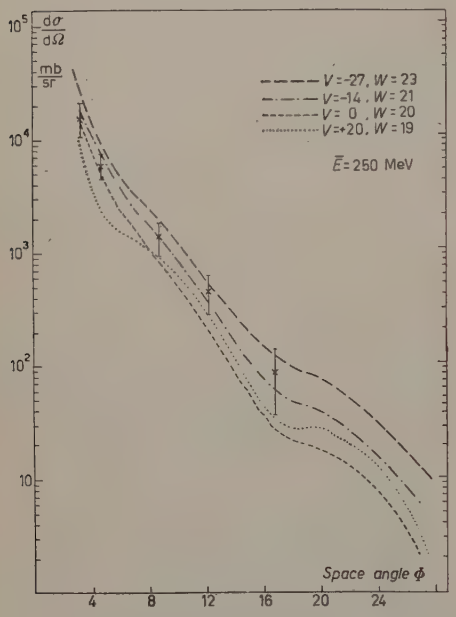

Fig. 5. – The differential cross-section for the elastic scattering of K^+-mesons from emulsion nuclei. The four curves are calculated exactly on the basis of the optical model using the parameters listed.

connection with low energy proton scattering from nuclei ([10]), on the electronic computer S.W.A.C. The fixed parameters used are essentially the same as those employed by IGO *et al.* ([11]) in an exactly similar calculation at a lower energy, namely $r_0=1.07$ fermi and $a=0.56$ fermi. The imaginary part of the potential, W, is determined substantially independently of V, the real part, by a comparison with the known reaction cross-section in emulsion $=420$ mb. The value of W required to fit this is $(19 \div 21)$ MeV for V between -20 and $+30$ MeV. In Fig. 5, four representative curves are shown corresponding to various values of V. A χ^2 analysis was performed on the results and the χ^2 probability is shown in Fig. 6. The distribution peaks at 60% at a potential of 13 MeV repulsive. This is considerably lower than the (27 ± 3) MeV obtained by IGO *et al.* at 150 MeV (*). It presumably results from stronger interaction in the attractive $T=0$ p-states at 250 MeV compared with 150 MeV. It may also be remarked that any correction for contamination of « inelastic events » in the elastic classification will tend to further reduce the potential. It is expected that there is some inelastic contamination below 25°

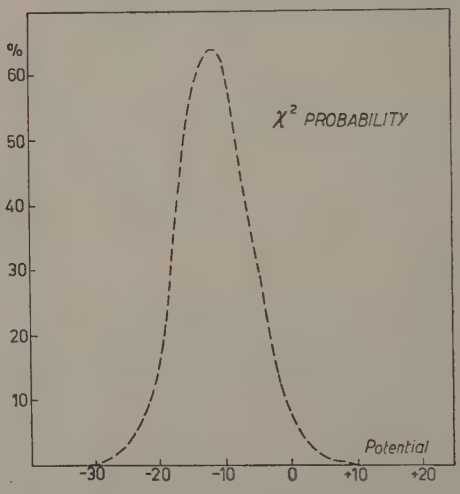

Fig. 6. – Values of the χ^2 probability as a function of the real optical potential.

because 3 events have been observed in which the K-meson was scattered through $\leqslant 25°$ and in which no energy loss would have been detected had not visible prongs been associated with the interaction. Further analysis is in progress

([11]) G. IGO, D. G. RAVENHALL, J. J. TIEMANN, W. W. CHUPP, S. GOLDHABER, G. GOLDHABER, J. E. LANNUTTI and R. M. THALER: *Phys. Rev.*, **109**, 2133 (1958).

(*) It must be mentioned that a similar analysis of some new data at 125 MeV gave a best value of $V=(21 \pm 4)$ MeV, MELKANOFF, PRICE, STORK and TICHO: private communication.

to correct for this effect. No elastic event has been observed in which the scattering angle was $\geqslant 25°$ in agreement with all the optical model curves for which the integrated cross-section $\geqslant 25°$ would correspond to less than 1 event in 90 m of track length.

5. – Discussion.

A mean free path of 53.9 cm implies a cross-section per bound nucleon of 8.3 mb to first approximation. To convert this into a cross-section per free nucleon, three corrections are necessary: that due to the Coulomb potential which reduces the cross-section in the ratio $(1 - V_c/E_K)$, that due to the Pauli exclusion principle which inhibits small angle scattering. This has been calculated by a number of authors (*e.g.* references [12-14] and [4]), it amounts to about 11% of the cross-section in this case. The third correction is the most important and it is the shading correction which corrects for the attenuation of the K-mesons through the nucleus. Because the m.f.p. is short this is a large correction and it is unfortunately very sensitive to the choice of radius. Using the curves calculated by ROSSI [13], one obtains a correction factor of ~ 1.9 for a square well radius parameter, r_0, of 1.25 fermi. The resultant cross-section per free nucleon is therefore ~ 18 mb. We may compare this with the value obtained directly via the optical model calculation on the reaction cross-section (cf. references [7] and [8]). This obviates the need for choosing an effective radius and for the shading correction as the optical model radius of 1.07 fermi is relatively well known (*). The mean free path in nuclear matter is related to W by the equation: $\Lambda = (\hbar c/2W)\beta$ where β is measured in the nucleus and in the K-nucleus C.M. system. Substituting the values obtained and using the 11% Pauli principle correction, the value of $\bar{\sigma}$, the mean free-nucleon cross-section is deduced to be 20 mb (+).

Although the data are of small statistical weight it is instructive to enquire as to the values and signs of the various phase shifts (×). There is no real evidence that the K-p cross-section is a function of energy, we therefore assume a constant value of 16 mb (obtained from the summary presented by S. GOLDHABER at the Padua-Venice Conference and the present high energy data). Assuming therefore no contribution from p-waves in the $T=1$ state, one obtains a δ_0' phase shift of $-35°$, the sign being chosen to fit the repulsive potential at low energies. Using the corrected charge exchange ratio, f, and the effective cross-section per free nucleon $\bar{\sigma}$, the δ_0^0 phase

(*) The nucleon density assumed in the optical model calculations follows the form of the Saxon potential and assuring the K-nucleon force to be short range we may have some confidence that the radius parameter, r_0, approximates to that found from electron scattering experiments: 1.07 fermi. The nucleon density is certainly more realistically represented by the Saxon potential than by the square-well which has been assumed in the calculation of the shading correction made above. Previous K-meson optical model calculations have also used these same parameters and a comparison, at least, is useful.

(+) This we consider a more reliable estimate than the value deduced using a shading correction for a square-well shape. That the two agree reasonably well may be taken as meaning that a square well radius of about 1.25 fermi is equivalent to a nucleon distribution following the Saxon potential with the parameters:

$$r_0 = 1.07 \text{ fermi} \quad \text{and} \quad a = .56 \text{ fermi.}$$

This « equivalent » square-well radius is unlikely to be constant with energy because of the variation of the m.f.p. in nuclear matter with energy.

(×) We are very grateful to Professor H. K. TICHO for his interest and for his help in this analysis.

(12) R. M. STERNHEIMER: *Phys. Rev.*, **106**, 1027 (1957).

(13) B. ROSSI: *High Energy Particles* (New York, 1952), chap. 7.

(14) E. HELMY, J. H. MULVEY, D. J. PROWSE and D. H. STORK: to be published, *Phys. Rev.* (1958).

shift deduced from the equation,

$$\frac{\sin \delta_0^0}{\sin \delta_0^1} \cos (\delta_0^1 - \delta_0^1) = \\ = \frac{1}{.55}\left[\left\{1 - 2f\right\}\frac{\bar{\sigma}}{\sigma_{\text{K-p}}} - .45\right],$$

is $\sim -8°$. The exact value is sensitive to the various experimental quantities but within reasonable error limits it is always small and generally repulsive. Limits can be set on the p-wave $T=0$ phase shifts by the angular distribution of the K-nucleon scattering and by the absolute cross-sections. Although the coefficients listed in Fig. 3 are not expected to be particularly reliable in view of all the secondary processes, the most noteworthy feature is the high ratio of the cross-sections at 180° and at 90° (*). To obtain a high ratio p-wave phase shifts have to be as large as possible and in general nearly equal and attractive. The magnitudes are limited by the total cross-sections however to about 25°. Thus one consistent set of phase shifts, supposing the p-wave $T=1$ interaction to be negligible, is: $\delta_0^1 = -35°$, $\delta_0^0 = -8°$ and δ_1^0 and $\delta_3^0 \lesssim +25°$.

They are also in reasonable agreement with the somewhat reduced overall repulsive potential of ~ 13 MeV deduced from the optical model although the derivation of the real part of the potential from the real parts of the forward scattering amplitudes for free nucleons is uncertain even at this energy. Again they are in accord with the trend of the data at low energies which can be explained by rather small attractive $T=0$ phase shifts.

Further analysis is in progress which includes among other refinements, allowance for different Pauli exclusion principle corrections on the K-p, K-n and K-n charge-exchange cross-sections; these are necessary in more accurate calculations because of the apparent strong forward peaking of the K-n charge-exchange cross-section which the above phase shifts predict. Our experimental data on the correlation between the angle and energy of the fast protons emitted from the interactions would support this conclusion.

* * *

We are greatly indebted to Dr. E. J. Lofgren and his colleagues for the exposures at the Bevatron. Thanks are due to Dr. C. Waller of Ilford Ltd for the preparation of the emulsion stacks. We wish to thank Dr. O. R. Price and Professor H. K. Ticho for many stimulating discussions and Professors M. A. Melkanoff and D. S. Saxon for much help and advice on the Optical model. D.E. and D.J.P. are grateful to their colleagues of the University of Padua for many informal discussions and for their kind hospitality in the fall of 1957. The contribution of Dr. A. Wataghin in the early stages of the experiment in Bristol is gratefully acknowledged.

(*) It has been shown [15] that the effect of double scattering (neglected here) is to significantly increase this ratio. Thus, when double scattering is taken into account, the above condition on the p-wave phase shifts may be expected to become less restrictive.

[15] O. R. Price, D. H. Stork and H. K. Ticho: *Phys. Rev.*, Lett. **1**, 212 (1958).

IL NUOVO CIMENTO Vol. X, N. 1 1° Ottobre 1958

An Investigation of Angular Correlations in the Decay of Charged Hyperons.

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(ricevuto il 19 Luglio 1958)

A collection of emulsion data (1) was presented at the Padua-Venice Conference, 1957, on possible asymmetries in the Σ-hyperon decay, due to the non-conservation of parity. For this purpose the hyperons were considered, which were produced in K -meson captures by nuclei. Among all these events a particular type of star was chosen, showing a fast π-meson and a Σ-hyperon and no other prongs (except an Auger electron or a blob in some cases). A large majority of such events should be due to the absorption of the K -meson by a bound proton giving rise to a charged π and a charged Σ, which emerge from the nucleus without interacting appreciably with the nucleus itself. Should the emission mechanism of the π's and the Σ's allow a sensible degree of polarization of the Σ's with respect to the plane defined by the momenta $\boldsymbol{P}_\Sigma$ and $\boldsymbol{P}_\pi$ and parity be not conserved in the Σ decay, asymmetries could exist in the angular distribution of the decay product with respect to the above mentioned plane (2).

Assuming these angular distributions to follow the $1+\alpha\cos\vartheta$ law, ϑ being defined by $\boldsymbol{P}_{\pi\text{-decay}}\cdot\boldsymbol{P}_\Sigma\times\boldsymbol{P}_\pi$, the following values of α were given at the Padua-Venice Conference, by collecting data of various laboratories.

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(1) Berkeley-Göttingen *Compilation, Padua-Venice Conference, 1957* (Mimeographed Report), (p. VII-5).

(2) B. T. Feld, *Padua-Venice Conference, 1957* Mimeographed Report, (p. XIX-14).

Decay mode	α
$\Sigma^+ \to p + \pi^0$ (at rest and in flight)	-0.29 ± 0.17
$\Sigma^+ \to n + \pi^+$ (at rest)	-0.33 ± 0.20
$\Sigma^\pm \to n + \pi^\pm$ (in flight)	-0.23 ± 0.24

By studying K^- nuclear interactions by means of four new stacks, exposed to the Heckman K^- beam of the Bevatron, a considerable number of events became available to us. They were selected and analysed in the same way previously described.

The distributions of the ϑ angles are given in the following tables:

TABLE I. – $\Sigma^+ \to p + \pi^0$ (*at rest and in flight*).

ϑ (c.m.s. of Σ)	$0° \div 60°$	$60° \div 90°$	$90° \div 120°$	$120° \div 180°$
No. of events	51.5	54	50.5	36

Up: 105.5 Down: 86.5

$\alpha = + 0.20 \pm 0.15$

TABLE II. – $\Sigma^+ \to n + \pi^+$ (*at rest*).

ϑ (c.m.s. of Σ)	$0° \div 60°$	$60° \div 90°$	$90° \div 120°$	$120° \div 180°$
No. of events	42.5	49.5	34	43

Up: 92 Down: 77

$\alpha = + 0.18 \pm 0.15$

TABLE III. – $\Sigma^\pm \to n + \pi^\pm$ (*in flight*).

ϑ (lab. system)	$0° \div 90°$	$90° \div 180°$
No. of events	40.5	30.5

$\alpha = + 0.28 \pm 0.26$

According to the whole of the data, it seems that the asymmetries, if at all existing, are likely to be smaller than the ones indicated by the previous compilation.

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Remarks about Gas Bubble Chamber Sensitivity.

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Istituto Nazionale di Fisica Nucleare - Sezione Aggregata di Genova

(ricevuto il 28 Luglio 1958)

1. – According to a model which satisfactorily explains a number of experimental results, bubbles are nucleated by an ionizing particle in those points of a superheated liquid or of a supersaturated gas-liquid solution, where a sufficiently large energy density gets localized, *i.e.* along δ-electron paths and, to a smaller extent, along the «core» of the particle path. Both in superheated liquid and supersaturated gas-liquid solution the bubble density along the path of a particle is strongly dependent on the thermodynamic characteristics of the system. In particular, the dependence law is, in the former case, a rising function of the temperature and superheating of the liquid and in the latter case, of the concentration and supersaturation of the solution.

The bubble density in tracks of particles of a well-known type is generally assumed to be an index of the sensitivity of a system (superheated liquid or supersaturated gas-liquid solution), this datum allowing the specific ionization rate to be evaluated. It is clear that the bubble growth process, from their nucleation to the observation time, can appreciably change the initial bubble number by the fusion of two or more bubbles. It follows, therefore, that the direct measurement of the bubble density cannot furnish, as a rule, an information showing correctly the specific ionization of the particle. The higher the sensitivity of a system to bubble nucleation, the more probable appears the fusion of two or more bubbles in the dynamical phase of growth. Sometimes this effect may be of such an importance as to destroy even remarkable differences (due to different ionization) in the number of nucleated bubbles.

One way to avoid the trouble is to choose properly the thermodynamic parameters of the system, so as to bring as close as possible to unity the ratio of the number of initial nuclei to the number of observed bubbles. One can approach this ideal condition by reducing (for a given optical apparatus and a suitable choice of the time-delay between the crossing particle and the flash light) the sensitivity of the system. In this way, the mean distance between two nucleated bubbles along the path of a particle may be large enough to allow them to grow up to a visible size, while keeping their individuality.

BLINOV, KRESTNIKOV and LOMANOV [1] have detected a 12.5% rise in the bubble density in tracks of electrons, when going from energies between 1 and 5 MeV to energies between 10 and 100 MeV. The effect was observed with a system of a rather poor sensitivity in a chamber well controlled as far as the operating stability is concerned.

The possibility of bringing out the relativistic rise in the energy loss of fast particles is very interesting in many respects. We can also assume this possibility as an index of merit for a system and its thermodynamical operating conditions.

From this point of view the supersaturated gas-liquid solutions should offer several advantages. Let us consider, for instance, the carbon dioxide-propane and ethane-propane systems, where the bubble growth is determined by the gas issuing from the solution and by the evaporating liquid. The bubble growing speed, as experimentally measured at room temperature, appears to be practically independent of the nature of the dissolved gas [2]. This fact leads to the assumption that the main contribution to the bubble growth is due to the boiling propane, once the value of its partial pressure in the mixture has reached or gone beyond during the expansion of the system. The gas contribution is believed, on the contrary, to be quite small, its actual function being to lower the surface tension of the mixture. Even at high concentrations, the gas contributes directly to the bubble growth by diffusing in the liquid, *i.e.* by a process relatively slow in itself. In the case of the two above-mentioned systems, the propane boils at room temperature and the bubble growth rate is therefore rather slow. A low value of the bubble growth rate results in a relatively long sensitivity time and in a rather acritical operation with regard to changes of temperature (within $(1 \div 2)$ °C) or of equilibrium pressure (a few tenths of an atmosphere).

These peculiar characteristics of the two investigated systems led us to conclude that they may allow the detection of the effect observed by BLINOV *et al.*, without seriously limiting the sensitivity conditions or requiring exceptional control methods (for operating temperature, initial and final pressure, etc.) in order to get a stable operation of the chamber.

The analysis of the data recently collected with the 1 liter gas bubble chamber exposed to the γ-ray beam of the Turin 31 MeV Betatron (*) led to the following results. The energy of about 120 electrons ($E > 5$ MeV) from Betatron γ-rays has been evaluated by multiple scattering measurements, and the number of bubbles per cm was counted on single tracks, limiting the count to the first part ($(3 \div 4)$ cm) of the track. The error in energy evaluation is not higher than

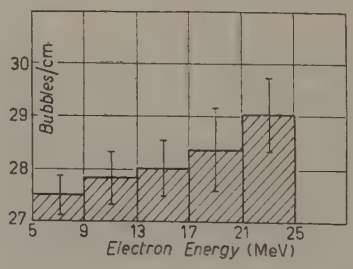

Fig. 1.

$(20 \div 30)$% for every scanned track. In Fig. 1 the mean bubble density in the tracks versus the electron energy scaled in 4 MeV steps is shown. The histogram refers to the carbon dioxide-propane

[1] G. A. BLINOV, IU. S. KRESTNIKOV and M. F. LOMANOV: *Soviet Phys., JETP*, **4**, 661 (1957).

[2] P. E. ARGAN, A. GIGLI, E. PICASSO, G. TOMASINI and L. GONELLA: *Report on Padua-Venice Conference* (Sept. 22-28, 1957).

(*) The methods used in exposition and data collection have been reported in detail at the Padua - Venice Conference - Sept. 1957 (see [2]).

system ($T=20$ °C; $P_{eq.}=29$ atm); the uncertainty in the bubble density is the statistical error. The tracks chosen for the present analysis lie with random distribution within a 20 ms time interval, *i.e.* about $\frac{2}{3}$ of the whole sensitivity time of the chamber.

Apart from some reserve about the statistical consistency of the data collected up-to-now, the histogram shows the sensitivity of the system to the relativistic rise in the energy loss of fast particles. There is a 6% effect for an energy variation from 5 MeV to 21 MeV.

Since the bubble growth features are practically independent of the nature of the dissolved gas, it is reasonable to expect that the same effect will be observed in every system where the propane is chosen as the liquid phase. A positive evidence of this statement was achieved with the ethane-propane system; the results were not plotted in Fig. 1 owing to the poor statistical data available up-to-now.

2. – Let us now examine the main characteristics of the bubble chamber described by BLINOV *et al.* in comparison with our ethane-propane chamber. They are reported in Table I; further on the two chambers will be referred to as *A* and *B*.

The operating conditions of chambers *A* and *B* are very different as far as the following points are concerned: *a*) operating temperature and its control; *b*) final pressure control; *c*) partial density of the hydrogen component; *d*) sensitivity to minimum ionizing particles. Other differences are less interesting.

The characteristics of chamber *B* offer remarkable advantages as compared to those of chamber *A*, and the numerical data brought to comparison in Table I do not require any comment. We want, however, to put forward a few critical remarks about some characteristics of chamber *A*.

Chamber *A* operates at 64 °C; the initial pressure is 38 atm and the final pressure, controlled within « a few hundredths of an atmosphere », is in three different cases 24.2, 23.5 and 22.8 atm (operating conditions I, II, III). The value of the vapor pressure quoted by BLINOV *et. al.* for the « singly distilled technical grade propane » which they used, is 30 atm at 64 °C, *i.e.* 7.4 atm higher than the pure propane vapor pressure (22.6 atm at 64 °C). The difference is easily explained by assuming that a certain amount of one or more gases is dissolved in the propane filling chamber *A*. If this is the case, the 30 atm value turns out to be the equilibrium pressure of a gas-

TABLE I.

Bubble Chamber type	Volume (litres)	Equilibrium pressure (atm)	Working pressure (atm)	Final pressure (atm)	Temperature (°C)	Density (g/cm³)	Hydrogen partial density (g/cm³)	Ethane mole fraction in the liquid phase	Bubble number per cm (min. ion. el.)	Total sensitivity time (ms)	Cycling time (s)
A	0.46	30	38	24.2 (I) 23.5 (II) 22.8 (III)	64 ± 0.1	0.425	0.077	—	2.4 2.5 14.0	40	10
B	1	25.6	30	> 6	room	0.465	0.090(*)	0.650	28	29	1

(*) This value was obtained by density and composition measurements made by us. The value of the density quoted in ref. (²) was only roughly estimated.

liquid solution at a temperature of 64 °C.

Let us tentatively suppose the dissolved gas to be methane or ethane. The equilibrium pressure of the methane-propane system will be 30 atm at 64 °C if the amount of the dissolved methane is 2% (expressed as molar fraction in the liquid phase), while for the ethane-propane system (at the same temperature and equilibrium pressure) the amount of dissolved ethane will be 13%. These data can be easily calculated from the common literature on the saturated hydrocarbons ([3]).

When the system is expanded from 30 atm to the final pressure of 22.8 atm it behaves typically as a supersaturated gas-liquid solution, and only during the last part of the expansion phase does the liquid evaporation take a remarkable part in the process of bubble growth. In operating conditions I and II, where the final pressure value is appreciably different from the propane partial pressure relative to the operating temperature, the contribution to bubble growth is due essentially to the gas, while the boiling propane contribution is much smaller.

On the basis of these arguments we can draw a few conclusions offering an easy interpretation of several peculiar operating features of chamber A, namely:

1) The need of stabilizing the final pressure within a few hundredths of an atmosphere in order to ensure the operating stability of the chamber, especially when the chosen final pressure value closely approaches the propane partial pressure.

2) The low sensitivity of the system to minimum ionizing particles, and its sharp dependence on the final pressure in proximity of the propane partial pressure value (see Fig. 3 of the quoted paper ([1])).

3) The long sensitivity time of the chamber.

4) The operating stability of the system, once its thermodynamic characteristic parameters are settled.

If our interpretation of the data reported by BLINOV *et. al.* is right, we can state in conclusion that chambers A and B operate in a *substantially identical way, i.e. both as gas bubble chambers*, with a single but strong difference in the couple of values taken for *concentration and temperature*.

When system A (low gas concentration and high temperature) is expanded from 30 atm to 22.8 atm. it behaves—in respect to ionizing radiation effects—as a supersaturated gas-liquid solution. By expanding the system to lower values of the final pressure, the effect due to the propane boiling at 64 °C would prevail, leading to a rise in the sensitivity and instability of the system.

When system B (high gas concentration and low temperature) is expanded from 25.6 atm to an uncontrolled final pressure, it behaves as a supersaturated gas-liquid solution until it reaches the value of the propane partial pressure. Below this value the effect due to propane boiling at (19÷20) °C comes to prevail. Chamber A is obviously more dependent on the stabilization requirements quoted at point 1). In chamber B the propane boils at room temperature, and therefore the stability of the system is not remarkably affected by the choice of the final pressure. It is apparent that the general stability features of chamber B could be further improved by applying the final pressure control methods adopted for chamber A.

The effect due to the relativistic rise in the ionization of fast electrons has been observed in both chambers A and B. In the former case this result was achieved by giving up a high sensitivity,

([3]) See for example: *Natural Gasoline Supply Men's Association; Engineering Data Book* (1951).

by imposing severe control conditions and by choosing the thermodynamic parameters in such a way as to operate the system, at least in part, as a supersaturated gas-liquid solution. In the latter case it was achieved though keeping a high sensitivity and without any special control condition on thermodynamic parameters (point 1 and 2). These differences and the characteristics remarked at point 3 and 4, which are typical of gas-liquid systems ([2]), can find their place in the general picture of the previous considerations.

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High Density and Large Atomic Number Systems for Gas Bubble Chambers.

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(ricevuto il 28 Luglio 1958)

The utilization of superheated liquids of relatively high density and large atomic number as working fluids in bubble chambers has been limited, up to now, by many serious experimental difficulties. On the other hand, systems having high stopping power and high γ-ray conversion efficiency are of great value for the study of several properties of elementary particles and, therefore, the realization of such a kind of bubble chambers, featuring good reliability and efficiency, is, at present, a very important problem.

The xenon chamber described by Brown, Glaser and Perl [1] shows no doubt exceptional characteristics in many respects, but its diffusion will be necessarily restrained by the high cost and difficult supply of xenon. More recently small bubble chambers (about 20 cm^3) filled with heavy liquids such as $SnCl_4$ and WF_6 were successfully operated. Both are corrosive liquids, and the temperatures at which they are sensitive to ionizing radiation are about 270 °C and 149 °C: this involves a number of such experimental difficulties (*e.g.* special processing required for materials, very careful preparation of liquids, etc.) which often give rise to serious doubts as to the reliability of these systems, especially for big chambers. The difficulties which arise in operating such systems are quite clearly shown in the recent paper by Alyea, Gallagher, Mullins and Teem [2].

Many difficulties are no doubt removed by giving up superheated liquids and by turning to systems based on different principles [3], such as supersaturated gas-liquid solutions, which can be radiation-sensitive at room temper-

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[1] J. L. Brown, D. A. Glaser and M. L. Perl: *Phys. Rev.*, **102**, 586 (1956).

[2] E. D. Alyea jr., L. R. Gallagher, S. H. Mullins and S. M. Teem: *Nuovo Cimento*, **6**, 1480 (1957).

[3] P. E. Argan, A. Gigli: *Nuovo Cimento*, **3**, 1171 (1956); **4**, 953 (1956).

ature, or — in any case — at not very high ones.

On the basis of this new idea ([3]) we have recently achieved some results by dissolving carbon dioxide in bromoethane (C_2H_5Br) and trichlorofluoromethane (CCl_3F). For densities of about $(1.3 \div 1.5)$ g/cm³ and not very large atomic numbers, these systems show good characteristics, though they lack the high reliability and efficiency required in experiments with big accelerators.

An interesting contribution to the study of two-component systems was given by HAHN and RIEPE ([4]), who successfully experimented fluorocarbon mixtures, in particular a mixture of ethforane (C_2F_6) and propforane (C_3F_8). This mixture shows a few remarkable properties and, as a system with no very high density and atomic number, it appears interesting from several points of view: the operating temperature is 21 °C, the equilibrium pressure 14 atm and density 1.4 g/cm³. Moreover, the saturated fluorocarbons are chemically stable, inert, and non-toxic.

There remains unsolved, however, the problem of working out high density-large atomic number mixtures which could offer, together with general characteristics comparable (excluding xenon for its exceptional characteristics) to those of pure $SnCl_4$ and WF_6, remarkable advantages in practical use and simplicity. This is actually possible and we intend showing it in this paper, by a *simple example.*

The liquid we used in the experiments is stannic chloride $SnCl_4$ (*). Its physical and chemical properties are well known: it is a transparent, colorless and fuming liquid, with 2.23 g/cm³ density at room temperature. It boils at 113.9 °C and freezes at — 33 °C; it is already strongly corrosive at room temperature. Carbon dioxide, chlorodifluoromethane and chlorotrifluoromethane are highly soluble in stannic chloride, without reacting chemically if handled with a certain care. The chemical and physical properties of these gases are well known, too.

Before experimenting in the bubble chamber, measurements were carried out on density, solubility and surface tension, as functions of the equilibrium pressure, on the following binary and ternary systems:

a) $SnCl_4 + CClF_3$;

b) $SnCl_4 + CHClF_2 + CO_2$;

c) $SnCl_4 + CHClF_2 + CClF_3$.

The binary mixture *a*) was prepared by dissolving directly the $CClF_3$ in the $SnCl_4$; the ternary mixture *c*) by dissolving in $SnCl_4$ first the $CHClF_2$, up to an equilibrium pressure of about 7.0 atm, then the other gas ($CClF_3$) up to the required final equilibrium pressure. The ternary mixture *b*) was prepared in a similar way.

The three mixtures were successively experimented in the 1 liter stainless-steel bubble chamber by which we recently investigated the properties of several systems with high hydrogen contents ([5]). The filling of the chamber was performed without substantially altering its original structure; just a few arrangements were necessary in order to avoid contamination of the mixture by water.

The chamber was exposed to the γ-ray beam of the Turin 31 MeV Betatron; the exposition procedure and the observation methods were the same we briefly described in a former paper ([5]).

([4]) B. HAHN, G. RIEPE: *Rev. Sci. Instr.*, **29**, 184 (1958).

(*) As already pointed out, $SnCl_4$ was investigated as a superheated liquid in small « clean » bubble chambers; in this case it becomes radiation-sensitive at about 261 °C. At 270 °C it may be used to visualize the paths of ionizing particles.

([5]) P. E. ARGAN, A. GIGLI, E. PICASSO, G. TOMASINI and L. GONELLA: *Report on the Padua-Venice Conference*, (Sept. 22-28, 1957).

TABLE I.

System	Volume of the chamber (cm^3)	Equilibrium pressure (atm.)	Temperature (°C)	Density (g/cm^3)	Bubble number/cm for minimum ionizing electrons	Radiation length (cm)	Total sensitivity time (ms)	Cycling time (s)
$SnCl_4$	20	20	270	1.4	(*)	8.6	(*)	(*)
WF_6	20	29	149	2.4	45÷50	3.8	1.5	10
$SnCl_4+CClF_3$	1000	28.5	room	1.7	25	9.1	> 30	1
$Sn_4Cl+CHClF_2+$ $+CClF_3$	1000	24.5	room	1.4	31	14.3	> 30	1

(*) unknown

Systems *a*) and *b*) become radiation-sensitive at concentrations corresponding to an equilibrium pressure of about 24 atm; system *c*) at about 19 atm. Systems *a*) and *c*) are sensitive to minimum ionizing electrons (Fig. 1) at equilibrium pressures of 28.5 (65 % $SnCl_4$ + 35 % $CClF_3$ by weight) and 24.5 atm (30% $SnCl_4$ + 15% $CHClF_2$ + 55% $CClF_3$ by weight).

The general characteristics of the chamber operating with systems *a*) and *c*) are summarized in Table I. For the sake of comparison, the table also reports the characteristics of the chamber described by ALYEA *et al.* [2] operating with superheated $SnCl_4$ and WF_6.

The parameter in which the four systems listed in the Table show the greatest difference, is the operating temperature. We need not point out in detail the advantages and the remarkable operational simplifications achieved in the work with corrosive liquids, when one can operate at room temperature. The equilibrium pressures of the three systems we investigated keep between 20 and 30 atm, which is an allowable range even for large volume chambers. The tungsten hexafluoride offers some advantages over the system *a*) in a few characteristic features (density and radiation length). It must be pointed out, however, that systems *a*) and *c*) have a long sensitivity time and are not critical to temperature variations: that involves a great operating stability, which is a typical feature of the gas bubble chambers.

The choice of a system, whether a superheated liquid or a supersaturated gas-liquid solution, is of course dependent on the particular physical problem one wants to study. The point we wish to emphasize is that, while the temperature is the main parameter one can work upon in order to get the best operating conditions of superheated liquids, in supersaturated solutions we can add to this possibility the choice (for a given liquid) of one or more gases and their concentrations in the solution. Such possibilities make these systems more widely suitable to peculiar experiments, and allow greater simplifications in several problems — *e.g.* when it is necessary to use corrosive or temperature-unstable liquids.

We have limited our present work to systems where the liquid phase is

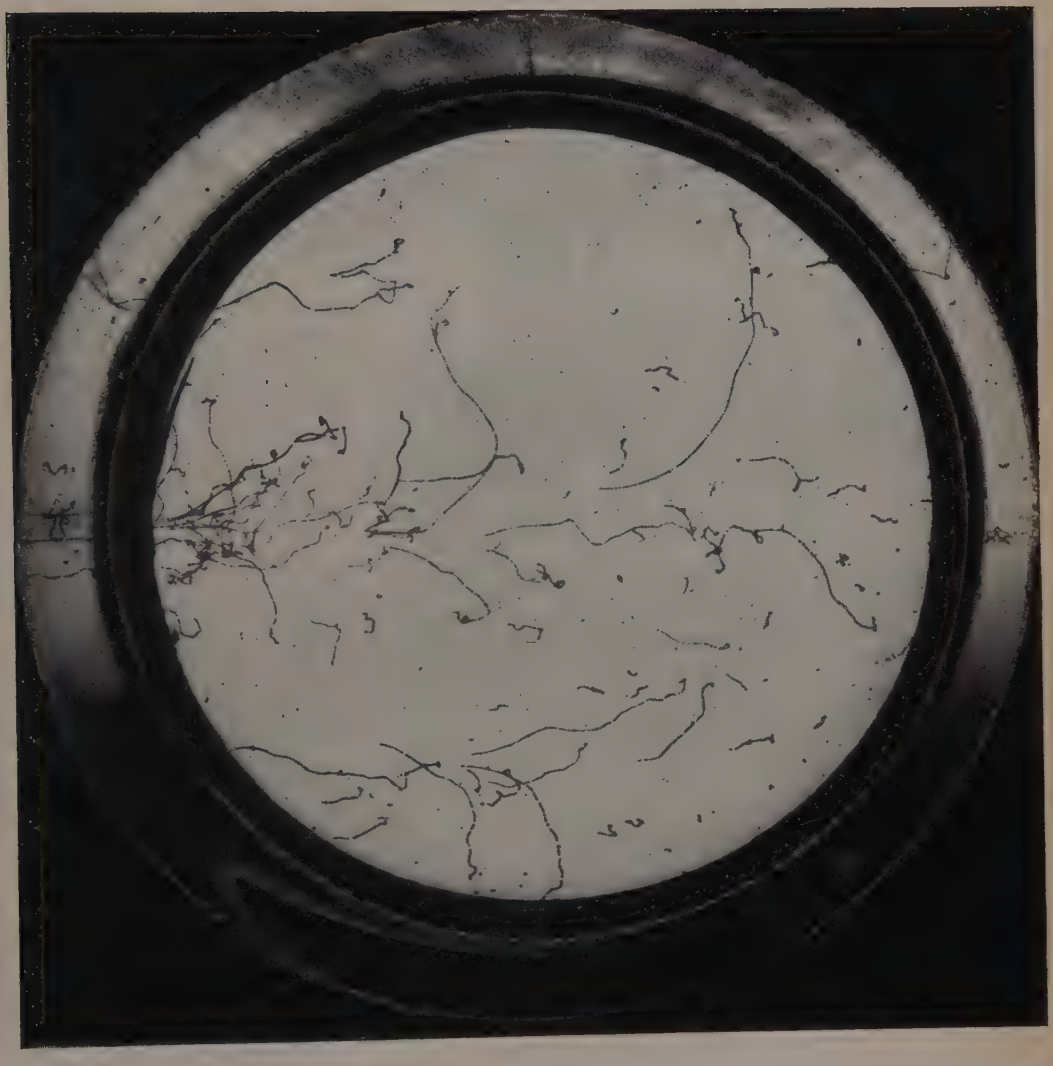

Fig. 1. – Photograph of the chamber taken after the expansion of the *c*) system from 24.5 to a few atmospheres. The flash light was pulsed one ms after the moment the collimated γ-ray beam of the Betatron crossed the chamber.

the stannic chloride; we essentially wanted to point out, by a *simple example*, the remarkable possibilities of the supersaturated gas-liquid solutions in the operation of bubble chambers of a special kind, with high efficiency and reliability. The data reported in Table I show by themselves these possibilities when compared with those concerning chambers working with superheated $SnCl_4$ or WF_6.

Other mixtures, featured by better characteristics and better suited to peculiar physical problems, may no doubt be utilized, keeping the operating temperature at room value or at a lower one in the case of corrosive and temperature-unstable liquids and gases (as for example $SnCl_4$, WF_6, PBr_3, $PbCl_4$), and at room value or at a higher one, in the case of non corrosive and rather stable liquids (as for example $CHBr_3$, CH_2I_2, C_2H_5I, C_3H_7I, CCl_3F and, possibly, $Hg(C_2H_5)_2$, $Hg(CH_3)_2$).

* * *

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K^--Nucleon Scattering with Tamm-Dankoff Approximation.

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(ricevuto il 9 Settembre 1958)

In a previous paper (1) it has been shown that the calculation of K^--nucleon scattering according to a Tamm-Dankoff approximation in which only the scattering diagram of Fig. 1 is considered (2), leads to completely wrong results, either if we assume scalar or pseudoscalar coupling for the K-Y-N interaction.

The main points that cannot be fitted with experimental data in this approximation are:

a) In the state $T=0$ which experimentally appears to be the most important one, the potential between K^- and nucleon turns out to be repulsive (for both scalar and pseudoscalar cases) while it should be attractive.

b) The experimental value of the cross-section (about 50 mb) cannot be reached with a value of the coupling constant of the same order as the interaction constant for K^+-N scattering; the calculation yielding thus values which cannot exceed $(10 \div 15)$ mb.

The possible reasons for such a failure were discussed in the paper quoted especially in connection with the somewhat better agreement that seems to be obtainable for the scattering of K^+-mesons on nucleons. It was concluded that, owing to the fact that the possible capture reaction of K^- is large at low energies, it is not justified to treat scattering independently from capture and that the Tamm-Dankoff calculation for K^- scattering should take into account the virtual capture diagrams which, in a two meson approximation, are naturally included in the solution of the complete Tamm-Dankoff equation.

Fig. 1.

In the present note, we have tried to develop, in a more systematic way, the solution for the complete Tamm-Dankoff equations for K^--nucleon pseudoscalar interaction which connect the K^--nucleon state with the Σ-π and Λ-π states according to the previous point of view, by separating, from the beginning, the different isotopic spin states.

Thus, at least for some given values of T and J (isotopic spin and total angular

(1) *Padua-Venice Conference* (Sept. 1957).
(2) D. Amati and B. Vitale: *Nuovo Cimento*, **6**, 261 (1957).

momentum), the equations are such that the Y-π amplitudes can be eliminated and one obtains a scattering equation for the K⁻-nucleon amplitude which can be solved according to the usual methods. This particularity happens for the $T=0$ and $l=0$ state, which, according to the actual experimental data obtained for K⁻-interaction in hydrogen, appears to be the principal scattering state at low energies. The equation for this state is then numerically solved and the results obtained are discussed.

The complete set of Tamm-Dankoff equations connecting the three K⁻-nucleon, Σ-π and Λ-π probability amplitudes $c_{NK}(\boldsymbol{k})$, $c_{\Sigma\pi}(\boldsymbol{k})$ and $c_{\Lambda\pi}(\boldsymbol{k})$ are the following ones:

$$
(1)\quad \begin{cases}
[E - M_N - \omega_{K^-}(\boldsymbol{k})]c_{NK^-}(\boldsymbol{k}) = \int d\boldsymbol{k}' c_{NK^-}(\boldsymbol{k}') K'_{NK^-}(\boldsymbol{k};\boldsymbol{k}') + \int d\boldsymbol{k}' c_{\Lambda\pi}(\boldsymbol{k}') K'_{\Lambda\pi}(\boldsymbol{k};\boldsymbol{k}') + \\
\qquad + \int d\boldsymbol{k}' c_{\Sigma\pi}(\boldsymbol{k}') K'_{\Sigma\pi}(\boldsymbol{k};\boldsymbol{k}') \,, \\
[E - M_\Lambda - \omega_\pi(\boldsymbol{k})]c_{\Lambda\pi}(\boldsymbol{k}) = \int d\boldsymbol{k}' c_{NK^-}(\boldsymbol{k}') K''_{NK^-}(\boldsymbol{k};\boldsymbol{k}') + \int d\boldsymbol{k}' c_{\Sigma\pi}(\boldsymbol{k}) K''_{\Sigma\pi}(\boldsymbol{k};\boldsymbol{k}') + \\
\qquad + \int d\boldsymbol{k}' c_{\Lambda\pi}(\boldsymbol{k}') K''_{\Lambda\pi}(\boldsymbol{k};\boldsymbol{k}') \,, \\
[E - M_\Sigma - \omega_\pi(\boldsymbol{k})]c_{\Sigma\pi}(\boldsymbol{k}) = \int d\boldsymbol{k}' c_{NK^-}(\boldsymbol{k}') K'''_{NK^-}(\boldsymbol{k};\boldsymbol{k}') + \int d\boldsymbol{k}' c_{\Sigma\pi}(\boldsymbol{k}') K'''_{\Sigma\pi}(\boldsymbol{k};\boldsymbol{k}') + \\
\qquad + \int d\boldsymbol{k}' c_{\Lambda\pi}(\boldsymbol{k}') K'''_{\Lambda\pi}(\boldsymbol{k};\boldsymbol{k}') \,,
\end{cases}
$$

where E is the total energy of the system and $\omega_{K^-}(\boldsymbol{k})$, $\omega_\pi(\boldsymbol{k})$ are the total energies of the K⁻ and π-meson respectively.

The different kernels K', K'' and K''' can be calculated with the usual standard methods and are not reported in this paper.

We will write, however, the esplicit form of the integral equations regarding only the $T=0$ and $l=0$ state on which we will fix our attention; we obtain in the case of pseudoscalar coupling:

$$
(2)\quad \begin{cases}
f^{00}_{NK^-}(\boldsymbol{k}) = -\dfrac{2g_\Lambda^2}{\pi} \dfrac{k^0_{K^-}\cdot \omega^0_{K^-}}{(\omega_{K^-}\cdot \omega^0_{K^-})^{\frac{1}{2}}(M_N + M_Y + \omega_{K^-})} + \\
\qquad + \dfrac{2g_\Lambda^2}{\pi} \int \dfrac{k'_{K^-}\cdot \omega'_{K^-}\, d\omega'_{K^-} f^{00}_{NK^-}(k')}{(\omega_{K^-}\cdot \omega'_{K^-})^{\frac{1}{2}}(\omega^0_{K^-} - \omega'_{K^-})(M_N + M_Y + \omega_{K^-} + \omega'_{K^-} - \omega^0_{K^-})} + \dfrac{\sqrt{6}\cdot g}{\pi} \cdot \\
\cdot \int \dfrac{k'_\pi \cdot \omega'_\pi \cdot d\omega'_\pi f^{00}_{\Sigma\pi}(k')}{(\omega_{K^-}\cdot \omega'_\pi)^{\frac{1}{2}}(M_Y + \omega'_\pi - M_N - \omega^0_{K^-})} \left[\dfrac{g}{2M_Y + \omega_{K^-} + \omega'_\pi - \omega^0_{K^-}} + \dfrac{g}{M_N + M_Y - \omega^0_{K^-}} \right], \\
f^{00}_{\Sigma\pi}(\boldsymbol{k}) = \dfrac{\sqrt{6}\cdot g}{\pi} \left\{ \dfrac{\omega^0_{K^-}\cdot k^0_{K^-}}{(\omega_\pi \cdot \omega^0_{K^-})^{\frac{1}{2}}} \left[\dfrac{g_\Lambda}{2M_Y + \omega_\pi} + \dfrac{g_\Sigma}{M_N + M_Y - \omega^0_{K^-}} \right] - \right. \\
\qquad \left. - \int \dfrac{\omega'_{K^-}\cdot k'_{K^-}\cdot d\omega'_{K^-} f^{00}_{NK^-}(\boldsymbol{k})}{(\omega_\pi \cdot \omega'_{K^-})^{\frac{1}{2}}(\omega^0_{K^-} - \omega'_{K^-})} \left[\dfrac{g_\Lambda}{2M_Y + \omega_\pi + \omega'_{K^-} - \omega^0_{K^-}} + \dfrac{g_\Sigma}{M_N + M_Y - \omega^0_{K^-}} \right] \right\} - \\
\qquad - \dfrac{g^2}{\pi} \int \dfrac{k'_\pi \cdot \omega'_\pi \cdot d\omega'_\pi f^{00}_{\Sigma\pi}(k')}{(\omega_\pi \cdot \omega'_\pi)^{\frac{1}{2}}(M_Y + \omega'_\pi - M_N - \omega^0_{K^-})} \cdot \\
\qquad \cdot \left[\dfrac{3}{3M_Y + \omega_\pi + \omega'_\pi - M_N - \omega^0_{K^-}} - \dfrac{1}{3M_Y - M_N - \omega^0_{K^-}} \right].
\end{cases}
$$

where the second equation of our set (1) which exists only for the $T=1$ states disappears entirely.

In (2) we have put $E=M_N+\omega^0_{K^-}$ and the mass difference between Σ and Λ particles has been neglected.

We may then solve the second integral equation (2) for the Σ-π amplitude in a Fredholm approximation and substitute the so obtained result in the first equation: we thus get the scattering equation for K^--nucleon amplitude in the form:

$$(3)\qquad f^{00}_{NK^-}(k) = -\frac{2g^2_\Lambda}{\pi}\frac{k^0_{K^-}\cdot\omega^0_{K^-}}{(\omega_{K^-}\cdot\omega^0_{K^-})(M_N+M_Y+\omega_{K^-})}+\frac{6g\cdot\omega^0_{K^-}\cdot k^0_{K^-}}{\pi^2(\omega_{K^-}\cdot\omega^0_{K^-})^{\frac{1}{2}}(1+\Delta_\pi)}I_1+$$
$$+\frac{2}{\pi}\int\frac{\omega'_{K^-}\cdot k'_{K^-}\cdot d\omega'_{K^-} f^{00}_{NK^-}(k')}{(\omega_{K^-}\cdot\omega'_{K^-})^{\frac{1}{2}}(\omega^0_{K^-}-\omega'_{K^-})}\left[\frac{g^2_\Lambda}{M_N+M_Y+\omega_{K^-}+\omega'_{K^-}-\omega^0_{K^-}}-\frac{3g}{\pi(1+\Delta_\pi)}I_2(k_{K^-};k'_{K^-})\right].$$

where:

$$(4)\qquad I_1(k_{K^-})=\int\frac{k'_\pi\cdot d\omega'_\pi}{(M_Y+\omega'_\pi-M_N-\omega^0_{K^-})}\left[\frac{g_\Lambda}{2M_Y+\omega_{K^-}+\omega'_\pi-\omega^0_{K^-}}+\frac{g_\Sigma}{M_N+M_Y-\omega^0_{K^-}}\right]\cdot$$
$$\cdot\left[\frac{g_\Lambda}{2M_Y+\omega'_\pi}+\frac{g_\Sigma}{M_N+M_Y-\omega^0_{K^-}}\right],$$

$$(5)\qquad \begin{cases} I_2(k_{K^-};k'_{K^-})=\displaystyle\int\frac{k'_\pi\cdot d\omega'_\pi}{(M_Y+\omega'_\pi-M_N-\omega^0_{K^-})}\left[\frac{g_\Lambda}{2M_Y+\omega_{K^-}+\omega'_\pi-\omega^0_{K^-}}+\frac{g_\Sigma}{M_N+M_Y-\omega^0_{K^-}}\right]^2, \\ \Delta_\pi=\dfrac{g^2}{\pi}\displaystyle\int\frac{k_\pi\cdot d\omega_\pi}{M_Y+\omega_\pi-M_N-\omega^0_{K^-}}\left[\frac{3}{3M_Y+2\omega_\pi-M_N-\omega^0_{K^-}}-\frac{1}{3M_Y-M_N-\omega^0_{K^-}}\right]. \end{cases}$$

This equation, solved with the same method as before, yields in first approximation:

$$(6)\qquad f^{00}_{NK^-}(\boldsymbol{k}) =$$
$$=\frac{1}{1+\Delta_K}\left[-\frac{2g^2_\Lambda}{\pi}\frac{k^0_{K^-}\cdot\omega^0_{K^-}}{(\omega_{K^-}\cdot\omega^0_{K^-})^{\frac{1}{2}}(M_N+M_Y+2\omega'_{K^-}-\omega^0_{K^-})}+\frac{6\omega^0_{K^-}\cdot k^0_{K^-}}{\pi^2(\omega^0_{K^-}\cdot\omega_{K^-})(1+\Delta_\pi)}I_1(k_{K^-})\right],$$

where:

$$(7)\qquad \Delta_K=\frac{2}{\pi}\int\frac{k'_{K^-}\cdot d\omega'_K}{\omega_{K^-}-\omega^0_{K^-}}\left[\frac{g^2_\Lambda}{M_N+M_Y+2\omega'_{K^-}-\omega^0_{K^-}}-\frac{3g}{\pi^2(1+\Delta_\pi)}I_1(k)\right].$$

In the preceeding formulae the semplifying hypothesis:

$$g_{\Sigma\Lambda\pi}=g_{\Sigma\Sigma\pi}=g_{N\pi}=g \quad \text{has been made.}$$

While the first term of (6) corresponds to the diagram of Fig. 1 (it can be obtained from the first equation (2) by putting $f_{\Lambda\pi}(k)=0$), the second term $I_1(k_{K^-})$ contains the contributions of the three diagrams of Fig. 2.

Numerical results of $(1+\Delta_K)\cdot f_{NK^-}(k^0_{K^-})$ and $1+\Delta_K$ are reported in Table I for two energies and for two values of the cut-off momentum.

TABLE I. – $g^2 = 15$, $g^2_\Lambda = 4{,}2$, $g^2_\Sigma = 1{,}4$.

	ω^0_K – in MeV	I	II	$(1+\Delta_K)\cdot f^{00}_{NK}(k^0_K)$	$1+\Delta_K$
$K = M_N$	50	– 0.2312	0.2202	– 0.0110	– 2.6713
	100	– 0.3286	0.3249	– 0.0037	– 2.1042
$K = M_Y$	50	– 0.2312	0.2272	– 0.0040	– 2.4974
	100	– 0.3286	0.3558	0.0272	– 2.0020

In the first column are indicated the results already calculated pertaining to the diagram of Fig. 1, which alone would give a negative phase shift and therefore a repulsive potential contrary to experience.

The results related to the new terms I_1 are indicated in the second column. They are of opposite sign from the preceding ones and of about the same order of magnitude and therefore contribute to correct the previous result in the right direction. This qualitative agreement indicating generally that higher order terms may be of such importance to compensate and eventually invert the sign of a pure scattering term is what we consider to be the positive result of our calculation.

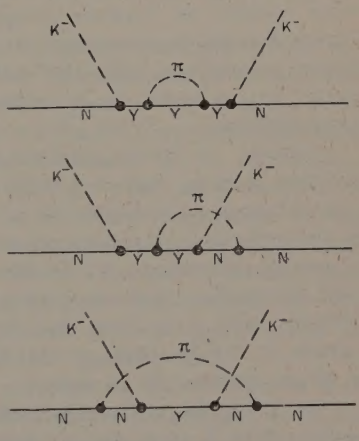

Fig. 2.

And this is probably the only fact that can be obtained at this stage and with the present means. In fact, if we except this qualitative point, there is not the least agreement in the qualitative aspect of the problem. The parameters we have chosen for the interaction constants and cut-off momentum according to what would appear to be reasonable values for them, are such that there is almost a complete compensation between the direct scattering negative terms and the new positive terms; small changes of the different constants and cut-off are likely to invert the signs as one likes, but not to increase the absolute value of the result in order to obtain fit with the observed data.

The only conclusion that we should like to draw from the present work is to stress that any calculation relative to the particular problem of scattering of K^- by nucleons has no meaning when only the direct interaction is considered, as it appears that even by remaining in the two meson approximation, there are surely other terms present so important as to reverse the sign of the results first obtained. As, however, the methods of calculation are such that no qualitative value for the different terms can be trusted, it appears that there is not much sense in trying to obtain numerical results to be compared with the experimental ones. This is the reason why we have not tried to push forward our calculation in order to derive values for scattering amplitudes for other isotopic spin and angular momentum states or for capture processes.

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Mesure de la radioactivité de l'air dans l'Antarctique (*).

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Détaché par l'Université Libre de Bruxelles auprès de l'Expédition Antarctique Belge 1957-1958

(ricevuto il 30 Settembre 1958)

Suivant les recommandations du Comité spécial de l'Année Géophysique Internationale, le programme de l'Expédition Antarctique Belge 1957-1958 comporte des mesures de la radioactivité atmosphérique.

Comme, à notre connaissance, il n'existe aucune donnée publiée sur la concentration en radioactivité artificielle et naturelle dans les régions antarctiques, il semble intéressant de donner ici un bref aperçu des résultats obtenus depuis Février 1958, à la Base Roi Baudouin (située à 70° 26′ Sud et 24° 19′ Est, à 10 km environ de la mer).

La méthode utilisée est classique. L'air aspiré à 10 m du sol passe avec un débit connu à travers un filtre retenant les particules de dimensions supérieures à 0.5 μm. L'activité β du filtre est mesurée pendant et après la filtration. L'analyse des courbes de croissance et de décroissance de l'activité permet de calculer les concentrations en radioéléments artificiels et naturels, liées aux poussières en suspension dans l'air.

Dans un petit nombre de cas, l'activité α des filtres a également été mesurée après la filtration. Les résultats publiés ici ont un caractère préliminaire, car la calibration exacte des détecteurs ne sera faite qu'après le retour de l'Expédition.

1. – Résultats.

1·1. *Radioéléments artificiels.* – L'activité β ayant une période supérieure à 10 jours a été attribuée aux produits de fission.

Cette activité a été mesurée d'une façon continue depuis le mois de février 1958. Elle s'est montrée extrêmement constante. Le rapport entre les mesures et la moyenne est compris entre 0.5 et 2. Les résultats sont:

moyenne	$2 \cdot 10^{-14}$ c/m^3,
minimum	$\sim 1 \cdot 10^{-14}$ c/m^3,
maximum	$4 \cdot 10^{-14}$ c/m^3.

Aucune corrélation n'a été observée avec les conditions météorologiques (température variant de -10 à -48 °C, vent variant de 0 à 60 noeuds).

1·2. *Radioéléments naturels.* – Les nuclides mesurés directement sont le ThB et le RaB; ce dernier peut être considéré comme étant en équilibre avec le Rn atmosphérique.

(*) Communiqué par radiotéléphonie, le 24 Septembre 1958.

Une activité α d'une période supérieure à 10 jours a été également détectée. Nous l'attribuons provisoirement au Po.

Les résultats sont:

$$\begin{aligned} \text{RaB} &< 10^{-12}\ \text{c/m}^3, \\ \text{ThB} &< 5\cdot 10^{-14}\ \text{c/m}^3, \\ \text{Po} &\quad 10^{-13}\ \text{c/m}^3. \end{aligned}$$

Les valeurs mesurées pour le RaB et le ThB ont été considérées comme des limites supérieures à cause d'une contamination possible par l'air intérieur des bâtiments de la base. Nous avons, en effet, trouvé dans l'atmosphère intérieure:

$$\begin{aligned} \text{RaB} &\quad 10^{-11}\ \text{c/m}^3, \\ \text{ThB} &\quad 4\cdot 10^{-12}\ \text{c/m}^3. \end{aligned}$$

Aucune trace d'éléments à longue période n'y a été détectée.

2. – Conclusions.

Une discussion de ces résultats serait tout à fait prématurée avant que ceux-ci n'aient été précisés et placés dans le cadre du réseau mondial des observations.

On remarquera cependant que la concentration en produits de fission est extrêmement faible par rapport à celle qui a été observée pendant la même période dans les latitudes tempérées de l'hémisphère nord. Le rapport des valeurs obtenues par nous à celles mesurées dans l'hémisphère nord est de l'ordre de 1/100.

Également remarquable est la constance de cette concentration, si on la compare aux variations d'un facteur 20 ou plus, couramment observées dans l'hémisphère nord.

Comme la quasi-totalité des produits de fission a été relâchée dans l'hémisphère nord, cette situation peut suggérer que l'échange est très faible entre la troposphère antarctique et l'ensemble de l'atmosphère au Nord de l'équateur; elle pourrait aussi indiquer que les masses d'air mesurées ici ont subi un processus très efficace de purification des poussières en suspension.

Les radioéléments naturels sont également présents en concentration très faible. Les limites supérieures trouvées sont mille fois plus faibles que les teneurs moyennes mesurées au-dessus des autres continents. Elles sont du même ordre de grandeur que les concentrations mesurées au-dessus des océans.

La concentration moyenne en U et en Th des roches antarctiques étant certainement du même ordre de grandeur que celle des roches des autres continents, l'absence de Rn dans l'air indique que le temps de diffusion du Rn à travers l'inlandsis est beaucoup plus long que 3.8 jours.

Une campagne systématique de mesures des radioéléments naturels dans l'atmosphère pourrait apporter des informations sur les échanges atmosphériques de part et d'autre de la convergence antarctique. En effet, la production de Rn et de Tn semble négligeable au Sud de la convergence antarctique (les seules sources étant l'inlandsis et l'océan), alors que les sources importantes les plus voisines sont les continents situés au Nord du 40^{e} parallèle sud.

La présence de Po et donc de RaD dans l'atmosphère antarctique ouvre des possibilités intéressantes pour la chronologie de la glace à l'échelle des cent dernières années, possibilités qui seraient un appoint précieux à celles offertes par le tritium.

Il est intéressant de noter que l'Antarctique semble être le continent de loin le moins radioactif. C'est aussi vraisemblablement la région du globe la moins affectée par les produits des explosions nucléaires, ce qui peut être très utile pour la continuation de certaines recherches de géochimie nucléaire.

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LIBRI RICEVUTI E RECENSIONI

N. M. GÜNTER – *Die Potentialtheorie und ihre Anwendung auf Grundaufgaben der Mathematischen Physik*. B. G. Teubner Verlagsgesellschaft, Leipzig, 1957.

Il volume è dedicato ad una trattazione esauriente e completa della teoria classica del potenziale, che viene esposta in forma piana e rigorosa. Il primo capitolo contiene una raccolta di teoremi ausiliari di Analisi (teorema di Hugoniot-Hadamard, teoremi di Gauss-Green e di Stokes, identità integrali, ecc.) che verranno poi impiegati nel seguito. Nel secondo capitolo trovano posto i risultati fondamentali relativi ai potenziali di semplice e doppio strato di una superficie ed a quello di volume. La teoria qui svolta viene, secondo i ben noti, classici procedimenti, applicata alla risoluzione dei problemi al contorno, interni ed esterni, per l'equazione di Laplace, nei successivi capitoli terzo e quarto. Particolarmente accurato lo studio delle proprietà qualitative della soluzione dei vari problemi al contorno. Il cap. V è dedicato alle funzioni di Green e di Neumann ed alle consuete applicazioni di queste per la soluzione di problemi al contorno relativi a particolari equazioni di tipo ellittico, interessanti la Fisica matematica. Chiudono la trattazione tre appendici dedicate allo studio di ulteriori proprietà qualitative dei potenziali di semplice e doppio strato.

Alla fine del volume si trova una biografia dell'autore, dovuta a W. I. SMIRNOV e S. L. SOBOLEV.

G. FICHERA

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